

Homework # 1

1. Let

$$\mathcal{A}_0 \doteq \{(a, b) \mid -\infty \leq a \leq b \leq \infty\} \cup \{\{a\} \mid -\infty < a < \infty\}.$$

Define

$$\mathcal{A} \doteq \{B_1 \cup B_2 \cdots B_k \mid k \in \mathbb{N}; B_i \in \mathcal{A}_0, i = 1, 2, \dots, k\}.$$

Show that \mathcal{A} is a field.

2. Let \mathcal{C}_1 and \mathcal{C}_2 be classes such that $\mathcal{C}_1 \subset \sigma(\mathcal{C}_2)$ and $\mathcal{C}_2 \subset \sigma(\mathcal{C}_1)$. Show that $\sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2)$.

3. Let Ω be a set and \mathcal{F} be a class of subsets of Ω . Suppose that $\Omega \in \mathcal{F}$ and that whenever $A, B \in \mathcal{F}$, we have that $A \cap B^c \in \mathcal{F}$. Show that \mathcal{F} is a field.

4. Let Ω be a set and $\mathcal{F}_1, \mathcal{F}_2, \dots$ be a sequence of fields of Ω . Suppose that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \geq 1$. Show that $\cup_{n=1}^{\infty} \mathcal{F}_n$ is a field.

5. Let Ω be a set and \mathcal{F} be a field of Ω . Let $H \subset \Omega$ be such that $H \notin \mathcal{F}$. Show that the field generated by $\mathcal{F} \cup \{H\}$ consists of sets of the form

$$(H \cap A) \cup (H^c \cap B), \text{ where } A, B \in \mathcal{F}.$$

6. Let Ω be an infinite set. Let \mathcal{F} be the class of sets which are either finite or co-finite (A set is called co-finite if its complement is finite). It is easy to check that \mathcal{F} is a field. Define a set measure μ on \mathcal{F} such that $\mu(A) \doteq 0$ if A is finite and $\mu(A) \doteq \infty$ if A is co-finite.

(a) Show that μ is finitely additive.

(b) Show that if Ω is countable then μ is not countably additive.

(c) Show that if Ω is uncountably infinite then μ is countably additive.