Homework # 11

1. Let T, $I\!R^T$ and $\mathcal{R}^T$, $\mathcal{R}_0^T$ be as in class. Show that
(a) $\mathcal{R}_0^T$ is a Field.

(b) $\sigma(\mathcal{R}_0^T) = \mathcal{R}^T$

Hint. Try to write the cylinder sets in terms of the canonical coordinate process.

2. Construct a probability space on which is defined an i.i.d. sequence of $N(0,1)$ random variables.

3. Let ν be a probability measure on $(I\!R, \mathcal{B}(I\!R))$. Also let $p : I\!R \times \mathcal{B}(I\!R) \to [0,1]$ be a function such that for every $x \in I\!R$, $p(x, \cdot)$ is a probability measure on $(I\!R, \mathcal{B}(I\!R))$ and for every $B \in \mathcal{B}(I\!R)$, $p(\cdot, B)$ is a measurable function from $(I\!R, \mathcal{B}(I\!R))$ to $(I\!R, \mathcal{B}(I\!R))$. Show that there is a sequence of random variables $\{X_n\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ such that for all $k \geq 1$ and $A_i \in \mathcal{B}(I\!R)$, $i = 1, \cdots, k$

$$P(X_1 \in A_1, \cdots X_k \in A_k) = \int_{A_1} \left(\int_{A_2} \left(\cdots \left(\int_{A_{k-1}} p(x_k, A_k) p(x_{k-1}, dx_k) \right) \cdots \right) p(x_1, dx_2) \right) \nu(dx_1).$$

Such a sequence of random variables is called a Markov chain with initial distribution ν and transition probability function $p(x, dy)$.

Hint. Recall the remark made in class regarding the simplified form of Kolmogorov Consistency Theorem for ordered sets.

4. Let ν and p be as in 3. Suppose that ν satisfies the condition

$$\int p(x, A) \nu(dx) = \nu(A), \quad \forall A \in \mathcal{B}(I\!R).$$

(We say that ν is an invariant measure for the Markov chain.) Show that there exists a sequence $\{Z_n\}_{n \in \mathcal{Z}}$, where $\mathcal{Z} \doteq \{\cdots - 1, 0, 1, \cdots\}$, defined on some probability space satisfying:

(a) $P(Z_n \in A) = \nu(A)$ for all $A \in \mathcal{B}(I\!R)$.
(b)

$$P(Z_n \in A_1, \cdots Z_{n+k} \in A_k) = \int_{A_1} \left(\int_{A_2} \left(\cdots \left(\int_{A_{k-1}} p(x_k, A_k) p(x_{k-1}, dx_k) \right) \cdots \right) p(x_1, dx_2) \right) \nu(dx_1),$$

for all $n \in \mathcal{Z}$ and $A_i \in \mathcal{B}(I\!R)$, $i = 1, 2, \cdots k$.

Hint. Same as problem 3.

5. Let $T = [0, \infty)$. Show that there exists a stochastic process $\{W_t\}_{t \in T}$ on some probability space satisfying:

(a) $W_t - W_s \sim N(0, (t-s))$ for all $0 \leq s < t < \infty$.

(b) $W(0) \equiv 0$.

(c) For all $k > 1$ and $0 \leq t_1 < t_2 < \cdots < t_k$, $(W_{t_1}, W_{t_2} - W_{t_1}, \cdots, W_{t_k} - W_{t_{k-1}})$ are independent random variables.

This is the first step in the construction of a Wiener process.

Hint. Try to write down what should be the measure induced by $(W_{t_1}, W_{t_2}, \cdots, W_{t_k})$.

6. Let $T = [0, \infty)$. A function $R : T \times T \to I\!R$ is called a positive definite kernel (also called a covariance kernel) if for all $k \geq 1$, $t_1, \cdots t_k \in T$ and $a_1, \cdots a_k \in I\!R \setminus \{0\}$,

$$\sum_{i=1}^{k} \sum_{j=1}^{k} a_i a_j R(t_i, t_j) > 0.$$

A stochastic process $\{X_t\}_{t \in T}$ is called a Gaussian process if all of its finite dimensional distributions are Gaussian, i.e. for all $k \geq 1$ and $t_1, \cdots t_k \in T$,

$$(X_{t_1}, \cdots X_{t_k})$$

is a multivariate Normal random variable.

Let R be a positive definite kernel. Show that there is a Gaussian process X_t defined on some probability space such that $E(X_t) = 0$ for all $t \in T$ and $cov(X_t, X_s) = R(t, s)$ for all $s, t \in T$.

Hint. Exactly the same hint as 5.