Homework # 12

1. Let μ, ν, ρ be σ- finite measures on $(\Omega, \mathcal{F})$.

(a) Show that if $\nu << \mu$ and $\mu << \rho$, we have that $\nu << \rho$ and

$$\frac{d\nu}{d\rho} = \frac{d\nu}{d\mu}\frac{d\mu}{d\rho}, \quad a.e.\ [\rho].$$

Hint. Show that $\int_A \text{l.h.s } d\rho = \int_A \text{r.h.s } d\rho$ for all $A \in \mathcal{F}$.

(b) Show that if $\nu \equiv \mu$ then $\frac{d\nu}{d\mu} = (\frac{d\mu}{d\nu})^{-1}$, a.e. μ (or ν).

(c) Suppose that $\nu << \rho$ and $\mu << \rho$ then

$$\frac{d(\mu+\nu)}{d\rho} = \frac{d\mu}{d\rho} + \frac{d\nu}{d\rho}, \quad a.e.\ [\rho].$$

2. Suppose that μ, ν, ν_n are finite measures on $(\Omega, \mathcal{F})$ and that $\nu(A) = \sum_{n=1}^{\infty} \nu_n(A)$ for all $A \in \mathcal{F}$. Let

$$\nu_n(A) = \int_A f_n d\mu + \nu_n'(A),$$

$$\nu(A) = \int_A f d\mu + \nu'(A)$$

be the decompositions into the absolutely continuous part and the singular part. Show that

(a) $f = \sum_{n=1}^{\infty} f_n$, a.e. μ.

(b) $\nu'(A) = \sum_{n=1}^{\infty} \nu_n'(A)$ for all $A \in \mathcal{F}$.

(c) $\nu << \mu$ iff $\nu_n << \mu$ for all n.

3. Let $\{\mu_n\}_{n\geq 1}$ be a sequence of finite measures on $(\Omega, \mathcal{F})$. Then there exists a finite measure μ on $(\Omega, \mathcal{F})$ such that $\mu_n << \mu$ for all n.

4. Let μ, ν be probability measures on $(\Omega, \mathcal{F})$. It is easy to see that $\mu - \nu$ is a signed measure. Recall the definition of $|\eta|$ for a signed measure η. Define

$$d_{TV}(\mu, \nu) \doteq \frac{|\mu - \nu|(\Omega)}{2}.$$

(a) Show that there exist $f, g \in L^1(\mu) \cap L^1(\nu)$ such that

$$(\mu - \nu)(A) = \int_A (f - g) d(\mu + \nu), \quad \forall\ A \in \mathcal{F}.$$

(b) Using (a) show that $d_{TV}(\mu, \nu) = d_{TV}(\nu, \mu)$.

(c) Using problem 3, show that for probability measures μ, ν, η,

$$d_{TV}(\mu, \eta) \leq d_{TV}(\mu, \nu) + d_{TV}(\nu, \eta).$$

(d) Show that $d_{TV}(\cdot, \cdot)$ is a distance on the space of all probability measures on $(\Omega, \mathcal{F})$.