Homework # 4

1. Let $\mathcal{F}$ be a σ- field on $I\!\!R$.

(i) Show that $\mathcal{B}(I\!\!R) \subset \mathcal{F}$ if and only if every continuous function from $I\!\!R$ to $I\!\!R$ is $\mathcal{F}\backslash\mathcal{B}(I\!\!R)$ measurable (i.e measurable from $(I\!\!R, \mathcal{F})$ to $(I\!\!R, \mathcal{B}(I\!\!R))$).

(ii) Define

$$\mathcal{F}_0 \doteq \sigma\{f^{-1}(B) | B \in \mathcal{B}(I\!\!R) \text{ and } f \in C(I\!\!R : I\!\!R)\},$$

where $C(I\!\!R : I\!\!R)$ is the set of all continuous functions from $I\!\!R$ to $I\!\!R$. Note that $\mathcal{F}_0$ is the σ-field generated by all continuous functions from $I\!\!R$ to $I\!\!R$. Show that $\mathcal{F}_0 = \mathcal{B}(I\!\!R)$.

[This shows that $\mathcal{B}(I\!\!R)$ is the smallest σ-field with respect to which all continuous functions are measurable.]

2. Let f_n, f be measurable maps from $(\Omega, \mathcal{F}) \to (I\!\!R, \mathcal{B}(I\!\!R))$. Let μ be a measure on $(\Omega, \mathcal{F})$. Suppose that there exists $A \in \mathcal{F}$ such that $\mu(A) < \infty$ and $f_n(\omega) \to f(\omega)$ for all $\omega \in A$. Show that there exists $B \in \mathcal{F}$ such that $B \subset A$, $\mu(B) < \epsilon$ and

$$\sup_{\omega \in A \backslash B} |f_n(\omega) - f(\omega)| \to 0$$

as $n \to \infty$. This is known as Egoroff's Theorem.

[Hint: Define

$$B_n^{(k)} \doteq \{\omega \in A | |f(\omega) - f_m(\omega)| < \frac{1}{k} \text{ for some } m \geq n\}.$$

Show that for all k, $B_n^{(k)} \downarrow \emptyset$ as $n \to \infty$. Now use continuity from above to get , for each fixed k, a n_k such that $\mu(B_{n_k}^{(k)}) < \frac{\epsilon}{2^k}$. Define $B = \bigcup_{k=1}^{\infty} B_{n_k}^{(k)}$. Check that B has the desired properties.]

3. A function $f : I\!\!R \to I\!\!R$ is said to be upper semi-continuous (u.s.c) at $x \in I\!\!R$ if for each $\epsilon > 0$ there is a $\delta > 0$ such that $f(y) < f(x) + \epsilon$ whenever $|x - y| < \delta$.

[This says that if $x_n \to x$ then $\limsup_{n\to\infty} f(x_n) \leq f(x)$.] Suppose that f is u.s.c at every $x \in I\!\!R$. Show that f is $\mathcal{B}(I\!\!R)\backslash\mathcal{B}(I\!\!R)$ measurable.

[Hint. Show that for every $M \in I\!\!R$, the set $f^{-1}[M, \infty)$ is closed.]

4. Do Exercise 2.2 (page 28) from the text book. (Text book is on reserve in the M/P library.)