

## Homework # 7

In all the problems below  $(\Omega, \mathcal{F}, \mu)$  is a measure space and  $\{f_n\}$ ,  $f, g$  are measurable functions. We say that a measurable function  $f$  is integrable if  $\int |f| d\mu < \infty$ .

1. Suppose that  $f \geq 0$  a.e. and  $\mu\{\omega : f(\omega) > 0\} > 0$ . Show that  $\int f d\mu > 0$ .
2. Suppose that  $f$  is integrable. Show that  $f = 0$  a.e., if and only if  $\int_A f d\mu = 0$  for all  $A \in \mathcal{F}$ .
3. Let  $\mu$  be a finite measure. Suppose that there exists a  $C \in (0, \infty)$  such that  $|f_n(\omega)| \leq C$ , a.e., for all  $n$ . Show that for all  $m > 0$ ,  $\int |f_n - f|^m d\mu \rightarrow 0$  as  $n \rightarrow \infty$ .
4. Let  $\mathcal{C}$  be a  $\pi$ -system containing  $\Omega$  and suppose that  $\sigma(\mathcal{C}) = \mathcal{F}$ . Suppose that  $f$  and  $g$  are integrable functions satisfying  $\int_A f d\mu = \int_A g d\mu$  for all  $A \in \mathcal{C}$ . Show that  $f = g$  a.e.
5. Suppose that  $f_n \geq 0$  and  $f \geq 0$  a.e. Suppose that for all  $n$ ,  $\int f_n d\mu = \int f d\mu = 1$ . Also suppose that  $f_n \rightarrow f$  a.e. Show that  $\int |f_n - f| d\mu \rightarrow 0$  as  $n \rightarrow \infty$ . This result is called Scheffe's Theorem.  
**Hint.** Define  $g_n = f - f_n$ . Note that  $\int |g_n| d\mu = 2 \int g_n^+ d\mu$  (why?). Now use DCT (dominated convergence theorem) to show that  $\int g_n^+ d\mu$  converges to 0.
6. Let  $f$  be integrable. Let  $\mathcal{C}$  be a field such that  $\sigma(\mathcal{C}) = \mathcal{F}$ . Show that to every  $\epsilon > 0$ , there exists a simple function of the form  $\sum_{i=1}^N x_i 1_{A_i}$  s.t. (a)  $A_i \in \mathcal{C}$  for all  $i$  and (b)  $\int |f - g| d\mu < \epsilon$ .  
**Hint.** Recall Theorem 1 from Homework 3.
7. Let  $f_n$  be integrable for all  $n$ . Suppose that  $\sup_n \int f_n d\mu < \infty$ . Further suppose that  $f_n$  increase to  $f$  a.e. Show that  $f$  is integrable and  $\int f_n d\mu \rightarrow \int f d\mu$ .  
**Hint.** First try to show that  $f$  is integrable and then use DCT. For the first part, use the fact that  $f_n^-$  decrease to  $f^-$ .