Homework # 8

In all the problems below $(\Omega, \mathcal{F}, \mu)$ is a measure space and $\{f_n\}$, f, g are measurable functions.

1. Let μ be a finite measure. Let $\{f_\lambda,\ \lambda \in \Lambda\}$ be a family of integrable functions. Show that the family is uniformly integrable if and only if: (i) $\sup_{\lambda\in\Lambda} \int |f_\lambda| d\mu < \infty$ and (ii) For every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{\lambda\in\Lambda} \int_A |f_\lambda| d\mu < \epsilon, \quad \text{whenever } \mu(A) < \delta.$$

2. Let μ be a probability measure. We will see later in class that if f is an integrable function then $\int |f| d\mu = \int_0^\infty \mu(|f| \geq y) dy$. Use this fact to show the following.

(a) If $\lambda \geq 0$ then

$$\int |f| 1_{|f|\geq\lambda} d\mu = \lambda\mu(|f| \geq \lambda) + \int_\lambda^\infty \mu(|f| \geq y) dy.$$

(b) Suppose that g is an integrable function such that

$$\mu(|f_n| \geq y) \leq \mu(|g| \geq y), \ \ \forall\, y \in (0, \infty),\ n \geq 1.$$

(We say that the sequence $|f_n|$ is stochastically bounded by g.) Show that the family $\{f_n\}$ is uniformly integrable.

3. Suppose that μ is a finite measure. Suppose that the sequence $\{f_n\}$ is uniformly integrable. Show that

$$\limsup_{n\to\infty} \int f_n d\mu \leq \int \limsup_{n\to\infty} f_n d\mu.$$

4. Suppose that $f_n \geq 0$ a.e. and $f \geq 0$ a.e. Also suppose that f_n and f are integrable. Define $\nu_n : \mathcal{F} \to I\!R_+$ and $\nu : \mathcal{F} \to I\!R_+$ as follows.

$$\nu_n(A) \doteq \int_A f_n d\mu, \ \ \nu(A) \doteq \int_A f d\mu.$$

Show the following:

(a) ν_n and ν are finite measures.

(b) Suppose that $f_n \to f$ a.e. and $\nu_n(\Omega) \to \nu(\Omega)$. Show that

$$\sup_{A\in\mathcal{F}} |\nu(A) - \nu_n(A)| \to 0 \ \text{ as } n \to \infty.$$

Hint. Imitate the proof of Scheffe's theorem in HW 7 (problem 5).

5. Suppose that $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ are sequences of measurable functions such that $f_n \to f$, $g_n \to g$, $h_n \to h$ a.e. Also suppose that $g_n \leq f_n \leq h_n$ a.e. Assume that all measurable functions in this problem are integrable. Finally suppose that $\int h_n d\mu \to \int h d\mu$ and $\int g_n d\mu \to \int g d\mu$. Show that

$$\int f_n d\mu \to \int f d\mu.$$