Homework # 9

Let $(\Omega, \mathcal{F})$ be a measure space. A set function

$$\nu : \mathcal{F} \to (-\infty, \infty),$$

is said to be a *signed measure* iff it has the countable additivity property. I.e.

$$\nu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \nu(A_i), \text{ if } A_i \in \mathcal{F} \text{ and } \ A_i A_j = \emptyset, \ i \neq j.$$

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let f be an integrable function. Show that

$$\nu(A) \doteq \int_A f d\mu, \quad A \in \mathcal{F} \tag{1}$$

is a signed measure.

The following is a well known theorem.
Theorem. (Jordan decomposition theorem) Let ν be a signed measure on $(\Omega, \mathcal{F})$. Then there exist sets $A^+, A^- \in \mathcal{F}$ such that
(a) $A^+ \bigcup A^- = \Omega$, $A^+ \cap A^- = \emptyset$.
(b) $\nu(B) \geq 0$ for all $B \subset A^+$ and $\nu(B) \leq 0$ for all $B \subset A^-$.

We will refer to A^+, A^- as the Jordan decomposition of Ω w.r.t ν.

Define ν^+, ν^- as

$$\nu^+(E) \doteq \nu(E \cap A^+), \quad \nu^-(E) \doteq -\nu(E \cap A^-), \quad E \in \mathcal{F}.$$

Clearly ν^+, ν^- and $|\nu| \doteq \nu^+ + \nu^-$ are finite measures. Also, $\nu(E) = \nu^+(E) - \nu^-(E)$. We refer to the measures ν^+, ν^- as the Hahn decomposition of ν.

2. Let ν be a signed measure on $(\Omega, \mathcal{F})$ and let A^+, A^- be a Jordan decomposition of Ω.

(a) Show that if B^+ and B^- is another Jordan decomposition of Ω w.r.t. ν and

$$\tilde{\nu}^+(E) \doteq \nu(E \cap B^+), \quad \tilde{\nu}^-(E) \doteq -\nu(E \cap B^-), \quad |\tilde{\nu}| \doteq \tilde{\nu}^+ + \tilde{\nu}^-, \quad E \in \mathcal{F}$$

then $\nu^+ = \tilde{\nu}^+$, $\nu^- = \tilde{\nu}^-$ and $|\nu| = |\tilde{\nu}|$. Also show that $|\nu|(A^+ \setminus B^+) = 0$. This shows that Jordan-Hahn decomposition is essentially unique.

(b) Show that for $E \in \mathcal{F}$, $\nu^+(E) = \sup_{F \subset E} \nu(F)$ and $\nu^-(E) = -\inf_{F \subset E} \nu(F)$.

(c) Show that if ν is as in (1) then

$$|\nu|(E) = \int_E |f| d\mu.$$

Hint. What is the Jordan decomposition of Ω w.r.t. ν?

Definition. **(a)** Two measures μ and ν on $(\Omega, \mathcal{F})$ are said to be mutually singular, if they have disjoint supports, i.e., there exist sets $S_\mu, S_\nu \in \mathcal{F}$ such that $\nu(\Omega \setminus S_\nu) = 0$, $\mu(\Omega \setminus S_\mu) = 0$ and $S_\mu \cap S_\nu = \emptyset$.
(b) A measure ν is said to be absolutely continuous with respect to a measure μ (and we write $\nu << \mu$), iff for every $A \in \mathcal{F}$ satisfying $\mu(A) = 0$, we have that $\nu(A) = 0$. If $\nu << \mu$ and $\mu << \nu$, we say that μ and ν are mutually absolutely continuous and we write $\mu \equiv \nu$.

3. Let μ, ν be measures on $(\Omega, \mathcal{F})$. Show that $\nu << \mu$ iff to every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\nu(A) < \epsilon \text{ whenever } \mu(A) < \delta.$$

4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that if $f \geq 0$ is a measurable function and $\nu(A) \doteq \int_A f d\mu$, then $\nu << \mu$.

5. Show that if ν is a signed measure then (i) $\nu^+ << |\nu|$, $\nu^- << |\nu|$ and (ii) ν^+ and ν^- are singular.