Homework 5.

Assigned Nov 8, 2005. Due Nov 22, 2005.

1. Let $a, b : [0, T] \to I\!R$ be meaurable maps satisfying

$$\int_0^T |a(t)|dt + \int_0^T b^2(t)dt < \infty.$$

It can be shown that

$$d\xi_t = a(t)\xi_t dt + b(t)dW(t), \;\; \xi_0 = x$$

admits a unique solution. Show by an application of Ito's formula that the solution is

$$Z_t \doteq \exp(\int_0^t a(s)ds)(x + \int_0^t b(s)\exp(-\int_0^s a(u)du)dW(s)).$$

2. Let the setting be as in the "Martingale Representation Theorem" presented in class. Let $Z \in \mathcal{M}_2$ be orthogonal to $\mathcal{M}_2^*$ and let $f : I\!R \to \mathbb{C}$ be bounded and measurable. Show that $\mathbb{E}(Z_t f(W_s)) = 0$ for all $s \leq t$.
Hint: First consider the special case $f(x) = e^{iux}$ and apply Ito's formula. You may also see Theorem 4.15 of K&S if you get stuck.

3. By applying Ito's formula obtain a stochastic evolution equation of the form

$$dX(t) = u(t)dt + v(t)dB(t),$$

where u and v are adapted (possibly multi-dimensional) processes satisfying appropriate integrability conditions, for the following choices of X.
(i) $X_t = 2 + t + \exp B_t$, where B is a 1-d BM.
(ii) $X_t = B_1^2(t) + B_2^2(t)$, where $B = (B_1, B_2)'$ is a 2-d sBM.
(iii) $X_t = (t_0 + t, B_t)'$ where B is a 1-d BM.
(iv) $X_t = (B_1(t) + B_2(t) + B_3(t), B_2^2(t) - B_1(t)B_3(t))$, $B = (B_1, B_2, B_3)'$ is a 3-d sBM.

4. Use Ito's formula to show that

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$

5. Use Ito's formula to show that the following are martingales.
(i) $X_t = e^{t/2}\cos(B_t)$
(ii) $X_t = (B_t + t)\exp(-B_t - t/2)$

6. In each of the following find the process f_t such that

$$F(\omega) = \mathbb{E}(F) + \int_0^T f(t)dB_t.$$

(i) $F = B_T$
(ii) $F = \int_0^T B_t dt$
(iii) $F = B_T^3$
(iv) $F = e^{B_T}$.