Practice Problems

1. Suppose that $X_n \Rightarrow X$ and X_n has a normal distribution with mean μ_n and variance σ_n^2. Show that X is normally distributed and $\mathbb{E}(X_n) \to \mathbb{E}(X)$, $\mathbb{E}(X_n^2) \to \mathbb{E}(X^2)$.

Hint. Use the fact that if e^{itb_n} converges (as $n \to \infty$) for all t in a neighborhood of 0, then b_n must converge to some $b \in \mathbb{R}$.

2. Show that if X_n and Y_n are independent for each n, X and Y are independent and $X_n \Rightarrow X$, $Y_n \Rightarrow Y$, then $X_n + Y_n \Rightarrow X + Y$.

[Hint. Use Skorohod's repn. theorem and a product space construction.]

3. Show that if ϕ is the ch.f. of a random variable X and $\lim_{t \downarrow 0}(\phi(t) - 1)/t^2 = c > -\infty$, then $E(X) = 0$ and $E|X|^2 = -2c < \infty$. In particular, if $\phi(t) = 1 + o(t^2)$ then $\phi(t) = 1$ for all t and $X = 0$ a.s.

4. If Y_n are r.v.s with ch.f.'s ϕ_n, then $Y_n \Rightarrow 0$ if and only if there is a $\delta > 0$ so that $\phi_n(t) \to 1$ for $|t| \leq \delta$.

5. Let $X_1, X_2, \cdots$ be i.i.d. with $\mathbb{E}(X_1) = 0$ and $var(X_1) \in (0, \infty)$. Let $S_n = X_1 + \cdots X_n$.

(a) Use Kolmogorov 0-1 law to show that $\mathbb{P}(\limsup S_n/\sqrt{n} > c)$ is either 1 or 0 for all $c > 0$.

(b) Use Central limit theorem to prove that the above probability is in fact 1 for all c and so $\mathbb{P}(\limsup S_n/\sqrt{n} = \infty) = 1$.

[Hint: Recall that for a sequence $\{\xi_n\}$ of r.vs $\{\limsup \xi_n > c\} \supseteq \limsup\{\xi_n > c\}$.]

(c) From Skorohod repn theorem, there are r.v's $\{Z_n\}, Z$ such that $Z_n \to Z$ a.s., Z_n has the same distribution as $S_n/\sqrt{n}$ and Z is $N(0,1)$ distributed. This suggests

$$\mathbb{P}(\limsup S_n/\sqrt{n} = \infty) = \mathbb{P}(\limsup Z_n = \infty) \ = \mathbb{P}(Z = \infty) = 0.$$

What is wrong with the argument?