Solutions of Homework # 9

1. For any positive real number K, let $N = \inf\{n : X_n \geq K\}$. Because that $\{X_n\}$ is a submartingale and N is a stopping time, $\{X_{n\wedge N}\}$ is a submartingale by applying the theorem in class.

   For any n, $X_{n\wedge N} \geq K + \sup_n \xi_n^+$, this leads to $E\sup_n X_{n\wedge N}^+ \leq E\sup_n \xi_{n\wedge N}^+ + K < \infty$

   By Martingale convergence theorem, $X_{n\wedge N}$ converges a.s. This implies X_n converges a.s., because that on $\{N = \infty\}$, $X_{n\wedge N} = X_n$.

   We know that
   $$\{X_n \text{ converges}\} \supseteq \{\sup_n X_n < \infty\}$$
   By assumption, $\sup_n X_n < \infty$ a.s. So we have
   $$P\{X_n \text{ converges}\} \geq p\{\sup_n X_n < \infty\} = 1$$
   hence the result holds

2. Let $Z_n = X_n - \sum_{i=1}^{n-1} Y_i$. It is easy to check Z_n is a supermartingale. For any $M \in (0,\infty)$, let $N = \inf\{k : \sum_{i=1}^{k} Y_i > M\}$, and N is a stopping time. Hence $\{Z_{n\wedge N}\}$ is also a supermartingale. Easy to check $Z_{n\wedge N} + M \geq 0$, thus $Z_{n\wedge N} + M$ converges almost surely to a finite limit. So does $Z_{n\wedge N}$.

   We have
   $$\{Z_n \text{ converges}\} \supseteq \{N = \infty\} = \{\sum_{i=1}^{\infty} Y_i \leq M\}$$
   Since $\sum_{n=1}^{\infty} Y_n < \infty$ a.s., $P\{Z_n \text{ converges}\} = 1$.

   Note $Z_n = X_n - \sum_{i=1}^{n-1} Y_i$, and $\sum_{i=1}^{\infty} Y_i < \infty$, this leads to X_n converges almost surely to a finite limit.

3. (i) $\mu << \nu$'s necessary and sufficient condition is
   $$\prod_{n=1}^{\infty}\{\sqrt{\alpha_n\beta_n} + \sqrt{(1-\alpha_n)(1-\beta_n)}\} > 0 \tag{1}$$
   (ii) Just need to show under the conditions, (1) is equivalent to $\sum(\alpha_n - \beta_n)^2 < \infty$. let $x_n(\alpha,\beta) = \sqrt{\alpha_n\beta_n} + \sqrt{(1-\alpha_n)(1-\beta_n)}$
   $$\prod_{n=1}^{\infty} x_n > 0 \Leftrightarrow \prod_{n=1}^{\infty}(1 - x_n) < \infty$$
   Note
   $$\begin{aligned} 2(1-x_n) &= 1 - \alpha_n + \alpha_n + 1 - \beta_n + \beta_n - 2\sqrt{\alpha_n\beta_n} - 2\sqrt{(1-\alpha_n)(1-\beta_n)} \\ &= (\sqrt{\alpha_n} - \sqrt{\beta_n})^2 + (\sqrt{1-\alpha_n} - \sqrt{1-\beta_n})^2 \\ &= (\alpha_n - \beta_n)^2\{\frac{1}{\sqrt{\alpha_n} + \sqrt{\beta_n}} + \frac{1}{\sqrt{1-\alpha_n} + \sqrt{1-\beta_n}}\} \end{aligned}$$

The conditions yield

$$2\sqrt{\varepsilon} \leq \sqrt{\alpha_n} + \sqrt{\beta_n} \leq 2\sqrt{1-\varepsilon}$$

and

$$2\sqrt{\varepsilon} \leq \sqrt{1-\alpha_n} + \sqrt{1-\beta_n} \leq 2\sqrt{1-\varepsilon}$$

these lead to

$$\frac{(\alpha_n - \beta_n)^2}{4(1-\varepsilon)} \leq 1 - x_n \leq \frac{(\alpha_n - \beta_n)^2}{4\varepsilon}$$

This implies the result.

4. (i) Checking the definition of martingale by using $E(Z_n) = E(E(Z_n|Z_{n-1}))$.

   (ii) $EZ_n = \mu^n \to 0$, this implies $Z_n = 0$ a.s. for all n sufficiently large. (Because if for any m, there exists an $n(m) > m$, s.t. $Z_{n(m)} \neq 0$, $EZ_{n(m)} \geq E1 = 1$, then the subsequence $Z_{n(m)}$ will not converges to 0, this leads a contradiction.)

   Then there exists an N, s.t. $Z_n = 0$ almost surely, for all $n \geq N$. this implies $Z_n/\mu^n \to 0$ a.s. as $n \to \infty$.