Solutions to HW3 STOR 635

#1 For any $\varepsilon > 0$

$$P\left(\frac{S_n - ES_n}{n} > \varepsilon\right) \le \frac{Var\left(\frac{S_n}{n}\right)}{\varepsilon^2} \le \frac{E(S_n)^2}{n^2\varepsilon^2} \le \frac{\sum_{i=1}^n E(X_i^2)}{n\varepsilon^2} \longrightarrow 0 \quad \text{as } n\to\infty$$

(C-S±)

$\left(\begin{array}{c} E(X_n^2)\to 0 \\ \text{Stoltz Thm} \end{array}\right)$

#2 $$E\left(\frac{S_n}{n} - \nu_n\right)^2 = \frac{1}{n^2} Var S_n = \frac{1}{n^2}\sum_{i=1}^n Var X_i \le \frac{1}{n}\sum_{i=1}^n \frac{Var X_i}{i} \longrightarrow 0 \quad \text{as } n\to\infty$$

(X_n uncorrelated)

$\left(\begin{array}{c} \frac{Var X_n}{n}\to 0 \\ \text{Stoltz Thm} \end{array}\right)$

$\Rightarrow \frac{S_n}{n} - \nu_n \xrightarrow{P} 0$ (Lecture Notes)

#3 $$P\left\{\left|\frac{X_1+\cdots+X_n}{n}\right| > \varepsilon\right\} \le \frac{E(X_1+\cdots+X_n)^2}{n^2\varepsilon^2} = \frac{n r(0) + 2\sum_{k=1}^{n-1}(n-k)\, r(k)}{n^2\varepsilon^2}$$

$$\frac{n r(0)}{n^2\varepsilon^2} = \frac{r(0)}{n\varepsilon^2} \to 0 \quad n\to\infty$$

$$\left|\frac{2\sum_{k=1}^{n-1}(n-k)\, r(k)}{n^2\varepsilon^2}\right| \le \frac{2}{\varepsilon^2}\frac{1}{n}\sum_{k=1}^{n-1}|r(k)| \longrightarrow 0$$

$\left(\begin{array}{c} |r(n)|\to 0 \\ \text{Stoltz Thm} \end{array}\right)$

$\Rightarrow \frac{X_1+\cdots+X_n}{n} \xrightarrow{P} 0$

#4 $U_1 \cdots U_n \cdots$ iid. & f measurable $\Rightarrow \{f(U_i)\}_{i=1}^{\infty}$ iid.

$\int_0^1 |f(x)|\,dx < \infty \Rightarrow E|f(U_1)| < \infty$

Note $Ef(U_1) = I$

$\Rightarrow I_n \xrightarrow{P} I$ (WLLN)

$$P\left[|I_n - I| > a/\sqrt{n}\right] \le \frac{Var\, I_n}{a^2/n} = \frac{Var\, X_1/n}{a^2/n} = \frac{Var X_1}{a^2} = \frac{\int_0^1 f^2(x)\,dx - \left(\int_0^1 f(x)\,dx\right)^2}{a^2}$$

#5 $P(X_1 > x)$ right continuous wrt x. & $\lim_{x \to e^+} P(X_1 > x) = 1$

$\Rightarrow P(X_1 \le e) = 0$

For any $M > 0$, choose $x = e^{\frac{e}{M}} > e$

$E|X_i| > x P(|X_i| > x) = x P(X_i > x) = e / \log e^{\frac{e}{M}} = M$

$\Rightarrow E|X_i| = \infty$

$\because x P(X_i > x) = \frac{e}{\log x} \to 0 \quad x \to \infty$

Choose $\mu_n = E[X_1 \mathbb{1}_{e < X_1 \le n}] \le n$

$\lim_n \mu_n = EX_1 = +\infty$ (Monotone Convergence Thm)

$\Rightarrow \frac{S_n}{n} - \mu_n \xrightarrow{P} 0$ (Lecture Notes)