Homework # 11

1. Let X, X_n, $n \geq 1$, be integer valued. Show that $X_n \Rightarrow X$ if and only if $I\!P(X_n = m) \to I\!P(X = m)$ for all m.

2. Let $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, where c is a constant. Show that:
(i) $Y_n \to c$ in probability.

(ii) $X_n + Y_n \Rightarrow X + c$. Consequently, if $Z_n - X_n \Rightarrow 0$ then $Z_n \Rightarrow X$.

(iii) $X_n Y_n \Rightarrow cX$.

Hint. Use Skorohod's rep. Th. and continuous mapping theorem.

3. Show that if $X_n = (X_n^1, \cdots, X_n^n)$ is uniformly distributed over the surface of the sphere of radius $\sqrt{n}$ in $\mathbb{R}^n$ then $X_n^1 \Rightarrow$ a standard normal r.v.

Hint. Use the fact that if $Y_1, \cdots$ are i.i.d. $N(0,1)$ then $(Y_1, \cdots Y_n)\left(n / \sum_{m=1}^n Y_m^2\right)^{\frac{1}{2}}$ has the same distribution as X_n.

4. Suppose g and h are continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $g(x) > 0$ for all x and $|h(x)|/g(x) \to 0$ as $|x| \to \infty$. Let F_n, F be distribution functions such that $F_n \Rightarrow F$ and $\int g(x) dF_n(x) \leq C < \infty$. Show that $\int h(x) dF_n(x) \to \int h(x) dF(x)$.

Hint. Use Skorohod repn. theorem and show that the family $\{h(\tilde{X}_n)\}_{n \geq 1}$ (where $\tilde{X}_n$ is the sequence in the cited theorem) is u.i.

5.
(i) Show that if $F_1, \cdots F_n$ are distribution functions with ch.fs $\phi_1, \cdots \phi_n$ resp. and $\lambda_i \geq 0$ are such that $\lambda_1 + \cdots \lambda_n = 1$ then the distribution function $\sum_{i=1}^n \lambda_i F_i$ has ch.f. $\sum_{i=1}^n \lambda_i \phi_i$.

(ii) Show that if ϕ is a ch.f. then $\Re\phi$ and $|\phi|^2$ are also, where for complex number z, $\Re z$ denotes its real part.