Homework # 2

1. Let $\{X_{ij},\ 1 \le i \le n,\ 1 \le j \le m(i)\}$ be a collection of independent random variables on a probability space $(\Omega, \mathcal{F}, I\!\!P)$. Let $f_i : \mathbb{R}^{m(i)} \to \mathbb{R}$ be measurable maps for $i = 1, \cdots n$. Let $U_i \doteq f_i(X_{i,1}, \cdots, X_{i,m(i)})$. Show that the random variables U_i, $i = 1, \cdots n$ are independent.

2. Recall that for a metric space S, $\mathcal{B}(S)$ denotes the Borel σ field. Show that $\mathcal{B}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R})^{\otimes n}$

3. Give examples of the following.

(i) Two random variables X and Y on a measurable space $(\Omega, \mathcal{F})$ and two probability measures P and Q on this space such that under P X, Y are independent while under Q they are not.

(ii) A probability measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R})^{\otimes 2})$ that is not a product measure.

(iii) A probability measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R})^{\otimes 2})$ that is a product measure.

4. Let $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F}$ the collection of all subsets of Ω and $P(\{i\}) = 0.25$, $i = 1, 2, 3, 4$. Give an example of two classes $\mathcal{A}_1, \mathcal{A}_2$ that are independent but whose generated σ fields are not.

5. Show that $X_1, \cdots X_n$ are independent if $\sigma\{X_1, \cdots, X_{k-1}\}$ is independent of $\sigma\{X_k\}$ for all $k = 2, 3, \cdots n$.

6. Let $\{X_i\}$ be a sequence of independent random variables on some probability space $(\Omega, \mathcal{F}, I\!\!P)$. Show that for any fixed n, $\sigma(X_1, \cdots X_n)$ is independent of $\sigma(X_{n+1}, X_{n+2}, \cdots)$.