Homework # 3

1. Let $\{X_n\}$ be a sequence of random variables given on some probability space. Suppose that $\mathbb{E}(X_n^2) \to 0$ as $n \to \infty$. Show that $\frac{S_n - \mathbb{E}(S_n)}{n}$ converges to 0 in probability as $n \to \infty$.

2. Let $\{X_n\}$ be a sequence of uncorrelated random variables with $\mathbb{E}(X_i) = \mu_i$ and $\text{Var}(X_i)/i \to 0$ as $i \to \infty$. Let $S_n = X_1 + \cdots X_n$ and $\nu_n = \mathbb{E}(S_n)/n$, then as $n \to \infty$, $S_n/n - \nu_n \to 0$ in L^2 and in probability.

3. Let $r : \mathbb{N}_0 \to (-\infty, \infty)$ be such that $r(k) \to 0$ as $k \to \infty$, where $\mathbb{N}_0$ is the space of nonnegative integers. Let $\{X_n\}$ be a sequence of random variables such that $\mathbb{E}(X_n) = 0$ and $\mathbb{E}(X_n X_m) \leq r(n-m)$ for $m \leq n$. Show that $(X_1 + \cdots X_n)/n \to 0$ in probability.

4. (**Monte Carlo Integration.**) Let f be a measurable function on $[0,1]$ with $\int_0^1 |f(x)|dx < \infty$. Let $U_1, U_2, \cdots$ be independent and uniformly distributed on $[0,1]$. Define

$$I_n \doteq n^{-1}(f(U_1) + \cdots f(U_n)).$$

Show that $I_n \to I \doteq \int_0^1 f(x)dx$ in probability. Suppose next that $\int_0^1 |f(x)|^2 dx < \infty$. Use Chebychev's inequality to estimate $\mathbb{P}(|I_n - I| > a/\sqrt{n})$.

5. Let $\{X_n\}$ be an i.i.d sequence with $P(X_i > x) = e/x \log(x)$ for $x > e$. Show that $\mathbb{E}|X_i| = \infty$ but there is a sequence of finite constants $\mu_n \to \infty$ so that $S_n/n - \mu_n \to 0$ as $n \to \infty$.