# Homework # 5

1. Let $X_n \geq 0$ be independent for $n \geq 1$. Show that the following are equivalent.
(i) $\sum_{n=1}^{\infty} X_n < \infty$ a.s.
(ii) $\sum_{n=1}^{\infty} (\mathbb{P}(X_n > 1) + \mathbb{E}(X_n 1_{X_n \leq 1})) < \infty$
(iii) $\sum_{n=1}^{\infty} \mathbb{E}(\frac{X_n}{1+X_n}) < \infty$.

2. Let $\psi(x) = x^2$ when $|x| \leq 1$ and equal to $|x|$ when $|x| \geq 1$. Show that if $X_1, X_2, \cdots$ are independent with $\mathbb{E}(X_n) = 0$ and $\sum_{n=1}^{\infty} \mathbb{E}(\psi(X_n)) < \infty$ then $\sum_{n=1}^{\infty} X_n$ converges a.s.

3. Let $X_1, X_2, \cdots$ be independent and let $S_{mn} = X_{m+1} + \cdots X_n$ for $m < n$ and define $S_{nn} = 0$. Show that

$$\mathbb{P}(\max_{m<j\leq n} |S_{mj}| > 2a) \min_{m<k\leq n} \mathbb{P}(|S_{k,n}| \leq a) \leq \mathbb{P}(|S_{mn}| > a). \tag{1}$$

4. Use (1) to show that if $X_1, X_2, \cdots$ are independent and $S_n = X_1 + \cdots X_n$, then $lim_{n\to\infty} S_n$ exists in probability implies that $lim_{n\to\infty} S_n$ exists a.s.

5. Let $X_1, X_2, \cdots$ be i.i.d. and $S_n = X_1 + \cdots X_n$. Use (1) to show that if $S_n/n \to 0$ in probability then $(\max_{1\leq m\leq n} S_m)/n \to 0$ in probability as well.