Homework # 9

1. Let $\{X_n\}_{n\geq 0}$ be a submartingale with $\sup X_n < \infty$ a.s. Let $\xi_n = X_n - X_{n-1}$ and suppose $\mathbb{E}(\sup_n \xi_n^+) < \infty$. Show that X_n converges a.s.
Hint. Follow the proof of the result on martingales with bounded increments that was covered in class.

2. Let X_n and Y_n be positive integrable and adapted to $\mathcal{F}_n$. Suppose that $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) \leq X_n + Y_n$ with $\sum Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit.
Hint. Introduce a supermartingale; consider the stopping time $N = \inf_k \sum_{m=1}^k Y_m > M$ and stop the supermartingale at time N. Also recall the proof for martingales with bounded increments.

3. Let $\mu, \nu, \hat{\mu}_n$ and $\hat{\nu}_n$ be as in the proof of Kakutani's theorem. Suppose that both $\hat{\mu}_n$ and $\hat{\nu}_n$ are concentrated on $\{0, 1\}$ for all n. Let $\alpha_n = \hat{\mu}_n\{1\}$ and $\beta_n = \hat{\nu}_n\{1\}$.
(i)Find a necessary and sufficient condition in terms of α_n, β_n for $\mu << \nu$.
(ii) Suppose that $0 < \epsilon \leq \alpha_n, \beta_n \leq 1 - \epsilon < 1$. Show that in this case the condition is simply $\sum(\alpha_n - \beta_n)^2 < \infty$.
Hint. Recall that $\prod_{m=1}^\infty x_m$ converges to a non zero quantity iff $\sum_{m=1}^\infty (1 - x_m)$ converges.

4. Let ξ_i^n, $i, n \geq 0$ be i.i.d. nonnegative integer valued random variables. Define a sequence Z_n, $n \geq 0$ by $Z_0 = 1$ and

$$Z_{n+1} = \xi_1^{n+1} + \cdots + \xi_{Z_n}^{n+1}, \text{ if } Z_n > 0.$$

If $Z_n = 0$, we set $Z_{n+1} = 0$. Let $\mathcal{F}_n = \sigma\{\xi_i^m : i \geq 1, 1 \leq m \leq n\}$ and $\mu = \mathbb{E}(\xi_i^m) \in (0, \infty)$.
(i) Show that Z_n/μ^n is a $\mathcal{F}_n$ martingale.
(ii) Show that, if $\mu < 1$ then $Z_n = 0$ for all n sufficiently large and so $Z^n/\mu^n \to 0$ as $n \to \infty$.