Homework # 1

1. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathbb{E}(|X|) < \infty$ and A_n is a sequence of events such that $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$. Show that $\lim_{n\to\infty} \int_{A_n} X d\mathbb{P} = 0$.

2. Given a random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite $\mathbb{E}(X)$, and $\epsilon > 0$, there exists a simple random variable X_ϵ such that $\mathbb{E}(|X - X_\epsilon|) < \epsilon$.

3. If F is a distribution function such that $F(0-) = 0$, then show that

$$\int_0^\infty (1 - F(x))dx = \int_0^\infty x dF(x) \leq +\infty.$$

Using this show that if $\{X_n\}$ is a sequence of i.i.d. random variables with finite mean, then

$$\lim_n \frac{1}{n} \mathbb{E}(\max_{1\leq j\leq n} |X_j|) = 0.$$

4. If $\mathbb{E}(X_1^-) < \infty$ and $X_n \uparrow X$ then $\mathbb{E}(X_n) \uparrow \mathbb{E}(X)$.

5. If $\mathbb{E}(|X|) < \infty$ and A_n are disjoint events with union A then

$$\sum_{n=0}^{\infty} \mathbb{E}(X 1_{A_n}) = \mathbb{E}(X 1_A),$$

i.e. the sum converges absolutely and has the value on the right.

6. Show that if $\mathcal{A}_1, \cdots, \mathcal{A}_n$ are independent π-systems then $\sigma(\mathcal{A}_1), \cdots \sigma(\mathcal{A}_2)$ are independent. (The case $n = 2$ was covered in class).