Representations for Functionals of Hilbert Space Valued Diffusions

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ABSTRACT: This paper contains two main results. The first is a variational representation for the expectation of a measurable function of a Hilbert space valued Brownian motion, when the function is uniformly positive and bounded from above and the Brownian motion has a trace class covariance. This representation is then applied to derive the second main result, which is the large deviation principle for a class of Hilbert space valued diffusions with small noise.

1 Introduction

The theory of large deviations is one of the classical topics in probability and statistics. For historical background and fundamental results in this area we refer the reader to Varadhan [24], Deuschel and Stroock [7], Dembo and Zeitouni [6], Freidlin and Wentzell [13], Ellis [12]. In a recent book [10] a new methodology was introduced for the analysis of large deviation problems. A crucial ingredient of this approach is the representation of the expectations whose asymptotic behavior is to be analyzed by value functions (minimal cost functions) of associated optimal stochastic control problems. With the representation in hand, one can then use weak convergence methods to study the convergence properties of the value functions as the large deviation scaling parameter tends to its limit. When rewritten in terms of the original expectations, one then obtains the desired large deviation result.

In the approach to large deviations just described, the variational representation for the pre-limit expectations is the starting point of the analysis. The canonical example of such a representation is the following (Proposition 1.4.2 [10]). Let $(\mathcal{V}, \mathcal{A})$ be a measurable space, k a bounded measurable function mapping $\mathcal{V}$ into $I\!R$ and θ a probability measure on $\mathcal{V}$. Then

$$-\log \int_{\mathcal{V}} e^{-k} d\theta = \inf_{\gamma \in \mathcal{P}(\mathcal{V})} \left\{ R(\gamma \| \theta) + \int_{\mathcal{V}} k d\gamma \right\}, \tag{1.1}$$

where $\mathcal{P}(\mathcal{V})$ is the space of all probability measures on $(\mathcal{V}, \mathcal{A})$ and $R(\cdot \| \cdot)$

denotes the relative entropy function (see Section 3 for the definition of relative entropy). Although this representation is very general (*e.g.*, it requires no topological properties on the underlying space), it must be rewritten in a more convenient form that reflects the structure of the underlying large deviation problem before it can be used in the convergence analysis. Many such refinements and various applications of the representations to the study of large deviation problems are in [10].

The main interest in the present work is the study of large deviations for infinite dimensional stochastic differential equations. The area of infinite dimensional stochastic calculus is a rapidly growing one, and various excellent references are available ([5], [25], [14], [17]). The "driving noise" in this calculus is an infinite dimensional (typically a Hilbert space valued) Brownian motion, and for this reason the underlying measure that plays the role of θ in (1.1) is Wiener measure on an infinite dimensional space. One of the main steps in our study is thus the derivation of a useful variational representation for expectations of exponential functionals of such a Brownian motion.

In a recent paper Boué and Dupuis [2] obtained the following representation for a finite dimensional Brownian motion. Let f be a bounded Borel measurable function mapping $\mathcal{C}([0,T]:\mathbb{R}^n)$ (the space of $\mathbb{R}^n-$valued continuous functions on $[0,T]$) into $\mathbb{R}$. Let W be a $\mathbb{R}^n-$valued standard Brownian motion. Then

$$-\log E\exp\{-f(W)\} = \inf_{v\in\mathcal{A}} E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 + f\left(W+\int_0^{\cdot} v(s)ds\right)\right), \tag{1.2}$$

where $\mathcal{A}$ is the space of square integrable progressively measurable processes. One of our main results (Theorem 3.6) is the extension of this representation to the case of infinite dimensional Wiener processes. Once a proper representation is available one can address a variety of large deviation questions for infinite dimensional stochastic differential equations. We illustrate the general methodology by studying the case of Hilbert space valued small noise diffusions in Section 4. The weakest conditions known to the authors on the drift and diffusion coefficients under which such diffusions have a unique strong solution are in Leha and Ritter [19]. We show under exactly these conditions that a large deviation principle (LDP) holds. As is the case for finite dimensional diffusions with such general coefficients contraction mapping arguments cannot be applied and the usual approach is via approximation (usually through a time discretization) of the original problem by a sequence of simpler problems. The LDP for each approximating problem is derived using a contraction mapping argument, and one finally obtains the LDP for the original problem by obtaining bounds on the approximation errors. In contrast, the approach presented here avoids such approximations and discretizations, which can be very useful in certain circumstances.

Small noise asymptotics of other classes of infinite dimensional stochastic differential equations have been studied by various authors. For example, the case of stochastic evolution equations in Hilbert spaces has been studied in [5], [22], [4]. Some references for the related case of reaction-diffusion stochastic partial differential equations are [26], [23], [16]. Although these classes of processes do not satisfy the assumptions made in Section 4, the representation proved in Section 3 is still applicable and can be used to establish the corresponding LDP. Unfortunately, space limitations prohibit the treatment of these cases in the present work.

The paper is organized as follows. In Section 2 we recall some facts about Hilbert space valued Brownian motions and weak convergence criteria for probability measures on Hilbert spaces. Section 3 is devoted to the proof of the main representation theorem. We also prove in this section a lemma (Lemma 3.1) concerning tightness of a certain sequence of Hilbert space valued processes. This lemma is used several times in rest of the paper. Finally in Section 4 we present the LDP for small noise diffusions in a Hilbert space.

One of the motivations for the present work is the study of image matching problems from a Bayesian perspective (*cf.* [11] and the references cited therein). Under appropriate regularity conditions on the coefficients, Hilbert space valued diffusions provide a natural class of prior models for the deformation that takes the canonical image into the target image suggested by the data. One potential application of large deviation theorems such as the one presented in Section 4 is to establish that the solution to the variational problem studied in [11] is an approximate maximum a posteriori estimator of the true deformation in the small noise limit.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \theta)$ be a probability space with an increasing family of right continuous θ-complete sigma fields $\{\mathcal{F}_t\}_{0\leq t\leq T}$. Let $(H, \langle \cdot, \cdot \rangle)$ be a real separable Hilbert space and let Q be a strictly positive, symmetric, trace class operator (*cf.* [9]) on H. The precise definition of an $H-$valued Wiener process with covariance Q can be found in [5, p. 87, 90]. For our purposes, the following two properties are essential:

1. For every nonzero $h \in H$, $\langle Qh, h\rangle^{-1/2}\langle W(t), h\rangle$ is a one dimensional standard $\mathcal{F}_t-$Wiener process.
2. For every $h \in H$, $W(t,h) \doteq \langle W(t), h\rangle$ is a $\mathcal{F}_t-$martingale.

Define $H_0 \doteq Q^{\frac{1}{2}} H$. Clearly, H_0 is a Hilbert space with the inner product

$$\langle h, k\rangle_0 \doteq \langle Q^{-1/2}h, Q^{-1/2}k\rangle,$$

$h, k \in H$. Denote the norms in H and H_0 by $\|.\|$ and $\|.\|_0$, respectively. Since Q is trace class the identity mapping from H_0 to H is Hilbert-Schmidt. This Hilbert-Schmidt embedding of H_0 in H will play a central role in many of the arguments to follow. One consequence of the embedding is that if $v^{(n)}$ is a sequence in H_0 such that $v^{(n)} \to 0$ weakly in H_0, then $\|v^{(n)}\| \to 0$. For an excellent exposition of stochastic calculus with respect to an $H-$valued Wiener process we refer the reader to [5]. Other useful references are [20], [21], [17].

The following two theorems are crucial ingredients of the proofs in this paper. Although the first theorem is standard (the reader may refer to [5, Theorem 10.14]), the second requires some elementary modifications of standard arguments and we refer the reader to [3] for the proof.

Let $\{\mathcal{G}_t\}_{0\le t\le T}$ be the $\theta-$completion of the filtration generated by $\{W(s) : 0 \le s \le t\}_{0\le t\le T}$. Define $\mathcal{A}$ to be the class of H_0-valued $\mathcal{F}_t-$progressively measurable processes ϕ that satisfy

$$\theta\left\{\int_0^T \|\phi(s)\|_0^2 ds < \infty\right\} = 1. \tag{1.3}$$

Finally, let

$$\mathcal{A}^W \doteq \{\phi \in \mathcal{A} : \phi \text{ is } \mathcal{G}_t - \text{progressively measurable}\}.$$

We refer the reader to [5, Chapter 4] for the definition of stochastic integrals of elements of $\mathcal{A}$ with respect to W.

Theorem 2.1 *Let $\psi \in \mathcal{A}$ be such that*

$$E\left(\exp\left\{\int_0^T \langle\psi(s), dW(s)\rangle - \frac{1}{2}\int_0^T \|\psi(s)\|_0^2 ds\right\}\right) = 1. \tag{1.4}$$

Then the process

$$\tilde{W}(t) \doteq W(t) - \int_0^t \psi(s)ds,$$

$t \in [0, T]$, *is a $Q-$ Wiener process with respect to $\{\mathcal{F}_t\}$ on $(\Omega, \mathcal{F}, \gamma)$, where γ is the probability measure defined by*

$$\frac{d\gamma}{d\theta} = \exp\left\{\int_0^T \langle\psi(s), dW(s)\rangle - \frac{1}{2}\int_0^T \|\psi(s)\|_0^2 ds\right\}.$$

Theorem 2.2 *Let $(M(t), \mathcal{G}_t)$ be a real valued local martingale with right continuous paths having left limits. Then there exists $\phi \in \mathcal{A}^W$ such that for all $0 \le t \le T$*

$$M(t) = M(0) + \int_0^t \langle\phi(s), dW(s)\rangle, \qquad a.s.$$

To finish this section we record two results that will be used in later sections to prove tightness for a sequence of Hilbert space valued processes. The first of these results is due to Aldous (*cf.* [25]). Let $(\mathcal{E}, d)$ be a Polish space. We denote by $\mathcal{C}([0,T] : \mathcal{E})$ the Polish space of continuous maps from $[0,T]$ to $\mathcal{E}$ equipped with the uniform topology. Although the tightness criteria in [25] is phrased in terms of processes with values in $\mathcal{D}([0,T] : \mathcal{E})$, the result as stated for processes in $\mathcal{C}([0,T] : \mathcal{E})$ follows as an immediate consequence (see [1] Chapter 3, Section 18, pages 150-153).

Theorem 2.3 *Let $\{X^{(n)}\}$ be a sequence of processes with paths in $\mathcal{C}([0,T] : \mathcal{E})$. Suppose that $\{X^{(n)}(t)\}$ is tight for each rational $t \in [0,T]$ and that for any sequence of stopping times $\{\tau_n\}$ such that $\tau_n \le T$ and any sequence of nonnegative numbers $\{\delta_n\}$ converging to zero as $n \to \infty$,*

$$d(X^{(n)}(\tau_n + \delta_n), X^{(n)}(\tau_n)) \to 0$$

in probability as $n \to \infty$. Then $\{X^{(n)}\}$ is tight.

The proof of the following theorem can be found in [17].

Theorem 2.4 *Let K be a separable Hilbert space and let $\{e_i\}$ be a complete orthonormal system (CONS) in K. Define P_N to be the projection operator with range equal to span$\{e_1, \cdots, e_N\}$. Let $\{\mu^{(n)}\}$ be a sequence of probability measures on $(K, \mathcal{B}(K))$. Then $\{\mu^{(n)}\}$ is tight if and only if*

1. *for all $N > 0$*

$$\lim_{A\to\infty} \sup_n \mu^{(n)} \left\{ x \in K : \max_{1\le i\le N} |\langle x, e_i\rangle| > A \right\} = 0,$$

2. *for any $\delta > 0$,*

$$\lim_{N\to\infty} \sup_n \mu^{(n)} \left\{ x \in K : \|x - P_N(x)\|_K \ge \delta \right\} = 0.$$

3 The representation theorem

This section is devoted to the proof of a representation theorem. For a bounded operator A on H let $\|A\|_{op}$ denote its operator norm. We begin with the following lemma.

Lemma 3.1 *Let $\{v^{(n)}\}$ be a sequence of elements of $\mathcal{A}$ (cf. (1.3)). Assume that*

$$M \doteq \sup_n \int_0^T E\|v^{(n)}(s)\|_0^2 ds < \infty. \tag{1.5}$$

Then the sequence $\left\{\int_0^{\cdot} v^{(n)}(s)ds\right\}$ is tight in $\mathcal{C}([0,T] : H)$.

Proof: For $0 \leq t \leq T$ define $X^{(n)}(t) \doteq \int_0^t v^{(n)}(s)ds$, and let $\{\tau_n\}$ and $\{\delta_n\}$ as in Theorem 2.3. The Cauchy-Schwarz inequality and the observation that $\|h\| \leq \|Q\|_{op}^{\frac{1}{2}} \|h\|_0$ for $h \in H_0$ imply

$$\|X^{(n)}(\tau_n + \delta_n) - X^{(n)}(\tau_n)\| \leq \sqrt{\delta_n}\, \|Q\|_{op}^{\frac{1}{2}} \left(\int_0^T \|v^{(n)}(s)\|_0^2 ds \right)^{1/2}.$$

Thus by (1.5) $\|X^{(n)}(\tau_n + \delta_n) - X^{(n)}(\tau_n)\|$ converges to 0 in $L^2(\theta)$. It now suffices, in view of Theorem 2.3, to show that for each fixed $t \in [0,T]$ the sequence $\{X^{(n)}(t)\}$ is tight in H. We will verify conditions 1 and 2 of Theorem 2.4. Let $\{\lambda_k\}$ be the sequence of eigenvalues of Q and let $\{e_k\}$ be a CONS of corresponding eigenvectors. Denote by $I\!N_0$ the set of positive integers. In order to verify condition 1, it suffices to note that for $A > 0$ and any $n, N \in I\!N_0$,

$$\theta \left\{ \max_{1 \leq i \leq N} |\langle X^{(n)}(t), e_i \rangle| > A \right\} \leq \theta \left\{ \sum_{i=1}^{N} \langle X^{(n)}(t), e_i \rangle^2 > A^2 \right\} \leq \frac{TM\, \|Q\|_{op}}{A^2}.$$

For condition 2, observe that if $f^{(n)}(s) \doteq Q^{-\frac{1}{2}} v^{(n)}(s)$ then the Cauchy-Schwarz inequality implies

$$\begin{aligned} \|X^{(n)}(t) - P_N(X^{(n)}(t))\|^2 &= \sum_{j=N+1}^{\infty} \left\langle \int_0^t v^{(n)}(s)ds, e_j \right\rangle^2 \\ &= \sum_{j=N+1}^{\infty} \left\langle \int_0^t f^{(n)}(s)ds, Q^{\frac{1}{2}} e_j \right\rangle^2 \\ &\leq T \int_0^T \|f^{(n)}(s)\|^2 ds \sum_{j=N}^{\infty} \lambda_i. \end{aligned}$$

Observing finally that $\|f^{(n)}(s)\| = \|v^{(n)}(s)\|_0$ and recalling (1.5), condition 2 is verified by an application of Chebychev's inequality. ■

The following lemma will be used in some of the tightness arguments in Sections 3 and 4.

Lemma 3.2 *Let $\{v^{(n)}\}$ be a sequence of elements of $\mathcal{A}$. Assume there is $M < \infty$ such that*

$$\sup_n \int_0^T \|v^{(n)}(s)\|_0^2 ds < M$$

a.s. Suppose further that $v^{(n)}$ converges in distribution to v with respect to the weak topology on $L^2([0,T] : H_0)$. Then $\int_0^{\cdot} v^{(n)}(s)ds$ converges in distribution to $\int_0^{\cdot} v(s)ds$ in $\mathcal{C}([0,T] : H)$.

Proof: For $N \in I\!\!N_0$ define

$$S_N \doteq \left\{ u \in L^2([0,T]:H_0) : \int_0^T \|u(s)\|_0^2 ds \leq N \right\}. \tag{1.6}$$

One can endow S_N with the weak topology, in which case it is a Polish space (*cf.* [9]). The lemma then follows immediately on observing that the map $\tau : S_N \to \mathcal{C}([0,T]:H)$ defined by $\tau(u) \doteq \int_0^\cdot u(s)ds$ is continuous. ■

The following lemma concerning measurable selections will be used in the proof of the main theorem below. We refer the reader to [3] for a proof.

Lemma 3.3 *Let E_1, E_2 be Polish spaces and let $f : E_1 \times E_2 \to I\!\!R$ be a bounded continuous function. Let K be a compact set in E_2. Define for each $x \in E_1$ the sets*

$$\begin{aligned} \Gamma_x^1 &\doteq \left\{ y \in K : \inf_{y_0 \in K} f(x,y_0) = f(x,y) \right\}, \\ \Gamma_x^2 &\doteq \left\{ y \in K : \sup_{y_0 \in K} f(x,y_0) = f(x,y) \right\} \end{aligned}$$

Then for $i = 1,2$ there exist Borel measurable functions $g_i : E_1 \to E_2$ such that $g_i(x) \in \Gamma_x^i$ for all $x \in E_1$.

For probability measures θ_1, θ_2 on $(\Omega, \mathcal{F})$ we define the relative entropy of θ_1 with respect to θ_2 by

$$R(\theta_1 \| \theta_2) \doteq \int_\Omega \left(\log \frac{d\theta_1}{d\theta_2}(\omega) \right) \theta_1(d\omega)$$

whenever θ_1 is absolutely continuous with respect to θ_2 and $\log(\frac{d\theta_1}{d\theta_2})$ is θ_1-integrable. In all other cases set $R(\theta_1 \| \theta_2) \doteq \infty$. Define $\mathcal{A}_N \doteq \{v \in \mathcal{A} : v(\omega) \in S_N \ \theta - a.s.\}$.

Lemma 3.4 *Let $\{f^{(n)}\}$ be a uniformly bounded sequence of real valued measurable functions on $\mathcal{C}([0,T]:H)$ converging to f a.s. θ. Then*

$$\inf_{v \in \mathcal{A}_N} E \left(\frac{1}{2} \int_0^T \|v(s)\|_0^2 + f^{(n)} \left(W + \int_0^\cdot v(s)ds \right) \right) \tag{1.7}$$

converges to

$$\inf_{v \in \mathcal{A}_N} E \left(\frac{1}{2} \int_0^T \|v(s)\|_0^2 + f \left(W + \int_0^\cdot v(s)ds \right) \right) \tag{1.8}$$

as $n \to \infty$.

Proof: Let $\epsilon > 0$ be arbitrary. For each $n \in I\!N_0$ pick an element $v^{(n),\epsilon}$ of $\mathcal{A}_N$ such that

$$E\left(\frac{1}{2}\int_0^T \|v^{(n),\epsilon}(s)\|_0^2 + f^{(n)}\left(W + \int_0^{\cdot} v^{(n),\epsilon}(s)ds\right)\right) \tag{1.9}$$

is at most ϵ larger than the infimum in (1.7). Since $\{v^{(n),\epsilon}, n \in I\!N_0\}$ is tight in S_N, we can pick a subsequence (relabeled by n) along which $(v^{(n),\epsilon}, W)$ converges weakly to (v^ϵ, W). Using Lemma 3.2 we have that $W + \int_0^{\cdot} v^{(n),\epsilon}(s)ds$ converges weakly as elements of $\mathcal{C}([0,T] : H)$ to $W + \int_0^{\cdot} v^\epsilon(s)ds$.

We next claim that

$$E\left(f^{(n)}\left(W + \int_0^{\cdot} v^{(n),\epsilon}(s)ds\right)\right) \to E\left(f\left(W + \int_0^{\cdot} v^\epsilon(s)ds\right)\right).$$

This is a consequence of [2, Lemma 2.8(b)], which states that for the last display to hold it is sufficient that the relative entropies

$$R\left(\mathcal{L}_\theta\left(W + \int_0^{\cdot} v^{(n),\epsilon}(s)ds\right) \|\mathcal{L}_\theta\left(W\right)\right)$$

be uniformly bounded in n, where $\mathcal{L}_\theta\left(W\right)$ and $\mathcal{L}_\theta\left(W + \int_0^{\cdot} v^{(n),\epsilon}(s)ds\right)$ denote the measures induced on $C([0,T] : H)$ by W and $W + \int_0^{\cdot} v^{(n),\epsilon}(s)ds$, respectively. But this is immediate since these relative entropies equal $E\int_0^T \|v^{(n),\epsilon}(s)\|_0^2 ds \leq N$. Using the weak convergence of $v^{(n),\epsilon}$ to v^ϵ and Fatou's lemma, it follows that

$$\liminf_{n\to\infty} E\left(\frac{1}{2}\int_0^T \|v^{(n),\epsilon}(s)\|_0^2 ds\right) \geq E\left(\frac{1}{2}\int_0^T \|v^\epsilon(s)\|_0^2 ds\right).$$

Thus the limit inferior, as $n \to \infty$, of the expression in (1.7) is at least the expression in (1.8).

For the reverse inequality, pick an element v^ϵ of $\mathcal{A}_N$ such that

$$E\left(\frac{1}{2}\int_0^T \|v^\epsilon(s)\|_0^2 ds + f\left(W + \int_0^{\cdot} v^\epsilon(s)ds\right)\right) \tag{1.10}$$

is at most ϵ larger than the infimum in (1.8). Clearly the expression in (1.10) with f replaced by $f^{(n)}$ is at least the infimum in (1.7). However, this quantity also converges to the expression in (1.10) as $n \to \infty$. This proves the reverse inequality and hence the lemma. ■

Lemma 3.5 *Let f be a bounded continuous function mapping $\mathcal{C}([0,T] : H)$ into $I\!R$.*

1. *Let* $\tilde{v} \in \mathcal{A}$ *be such that*

$$E\left(\exp\left\{\int_0^T \langle \tilde{v}(s), dW(s)\rangle - \frac{1}{2}\int_0^T \|\tilde{v}(s)\|_0^2 ds\right\}\right) = 1,$$

define $\tilde{W}(t) \doteq W(t) - \int_0^t \tilde{v}(s)ds$, *and let* $E^{\tilde{v}}$ *denote expectation with respect to the measure* $\gamma^{\tilde{v}}$ *defined by*

$$d\gamma^{\tilde{v}} \doteq \exp\left\{\int_0^T \langle \tilde{v}(s), dW(s)\rangle - \frac{1}{2}\int_0^T \|\tilde{v}(s)\|_0^2 ds\right\} d\theta.$$

Let $v_0 \in \mathcal{A}$ *be an elementary process such that for some* $M_0 \in (0,\infty)$ $\|v_0(s)\|_0 \leq M_0$ *a.s., for all* $s \in [0,T]$. *Then for every* $\epsilon > 0$ *there exist elementary processes* $v_1, v_2 \in \mathcal{A}^W$ *such that for* $i = 1,2$ $\|v_i(s)\|_0 \leq M_0$ *for all* $s \in [0,T]$, *and*

$$\begin{aligned} & E\left(\frac{1}{2}\int_0^T \|v_1(s)\|_0^2 ds + f\left(W + \int_0^{\cdot} v_1(s)ds\right)\right) - \epsilon \\ \leq\ & E^{\tilde{v}}\left(\frac{1}{2}\int_0^T \|v_0(s)\|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} v_0(s)ds\right)\right) \qquad (1.11) \\ \leq\ & E\left(\frac{1}{2}\int_0^T \|v_2(s)\|_0^2 ds + f\left(W + \int_0^{\cdot} v_2(s)ds\right)\right) + \epsilon. \end{aligned}$$

2. *Let* $\mathcal{A}^{(b)}$ *denote the subclass of* $\mathcal{A}$ *consisting of bounded elementary processes. Then*

$$\begin{aligned} & \inf_{\tilde{v}\in\mathcal{A}^{(b)}} E^{\tilde{v}}\left(\frac{1}{2}\int_0^T \|\tilde{v}(s)\|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} \tilde{v}(s)ds\right)\right) \\ =\ & \inf_{v\in\mathcal{A}} E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + f\left(W + \int_0^{\cdot} v(s)ds\right)\right). \end{aligned}$$

Proof: For the proof of part 1 we will use Lemma 3.3. We will only show the first inequality in (1.11) since the proof of the second inequality is similar, save that the corresponding supremization part of Lemma 3.3 is used instead. Suppose that the elementary process v_0 takes the form

$$v_0(s,\omega) \doteq X_0(\omega)\mathcal{I}_{\{0\}}(s) + \sum_{j=1}^{l} X_j(\omega)\mathcal{I}_{(t_j,t_{j+1}]}(s),$$

where $(s,\omega) \in [0,T]\times\Omega$, $0 = t_1 \leq t_2 \ldots \leq t_{l+1} = T$ and X_j are $H-$valued $\mathcal{F}_{t_j}$-measurable random variables satisfying $\|X_j(\omega)\|_0 \leq M_0$ a.s. for all

$j \in \{0, \dots l\}$. There is a continuous function $F_1 : H_0^{\otimes l+1} \to I\!R$ such that $F_1(X_0(\omega), \dots, X_l(\omega)) = \frac{1}{2}\int_0^T \|v_0(s)\|_0^2 ds$, a.s. Now define for $j = 1, \dots, l$ measurable maps $\tilde{Z}_j$ from Ω to $\mathcal{M}_j \doteq C([0, t_{j+1} - t_j], H)$ by

$$\tilde{Z}_j(\omega)(s) \doteq \tilde{W}(\omega)(s + t_j) - \tilde{W}(\omega)(t_j), \qquad 0 \le s \le t_{j+1} - t_j.$$

From the continuity and boundedness of the map f it follows that there exists a continuous bounded map $F_2 : \left(H_0^{\otimes l+1} \times \left(\prod_{j=1}^{l} \mathcal{M}_j\right)\right) \to I\!R$ such that

$$f\left(\tilde{W} + \int_0^{\cdot} v_0(s)ds\right) = F_2(X_j, 0 \le j \le l; \tilde{Z}_j, 1 \le j \le l),$$

a.s. For $1 \le i \le l$, let $\mathbf{X}_i$ and $\tilde{\mathbf{Z}}_i$ denote the vectors $(X_0, \dots, X_i)$ and $(\tilde{Z}_1, \dots, \tilde{Z}_i)$, respectively. With this notation

$$E^{\tilde{v}}\left(\frac{1}{2}\int_0^T \|v_0(s)\|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} v_0(s)ds\right)\right) = E^{\tilde{v}}\left(F_1(\mathbf{X}_l) + F_2(\mathbf{X}_l, \tilde{\mathbf{Z}}_l)\right). \tag{1.12}$$

We recall that every probability measure on a Polish space is tight. This implies there is a compact set $K_0 \subset H_0$ such that

$$E^{\tilde{v}}\left(F_1(\mathbf{X}_l) + F_2(\mathbf{X}_l, \tilde{\mathbf{Z}}_l)\right) \ge E^{\tilde{v}}\left(\mathcal{I}_{K_0^{\otimes l+1}}(\mathbf{X}_l)\left(F_1(\mathbf{X}_l) + F_2(\mathbf{X}_l, \tilde{\mathbf{Z}}_l)\right)\right) - \epsilon/4(l+1).$$

Since $(\tilde{W}(t), \mathcal{F}_t)$ is a Wiener process under $\gamma^{\tilde{v}}$, if $0 \le u_1 \le u_2 \le T$ then $\tilde{W}(u_2) - \tilde{W}(u_1)$ is independent of $\mathcal{F}_{u_1}$. Therefore, $\tilde{Z}_j$ is independent of $\left(\mathbf{X}_j, \tilde{\mathbf{Z}}_{j-1}\right)$ under $\gamma^{\tilde{v}}$. Let μ_j denote the standard Wiener measure on $\mathcal{M}_j$ and let $F_2^{(1)}$ be the real valued continuous map on $\left(H_0^{\otimes l+1} \times \left(\prod_{j=1}^{l-1} \mathcal{M}_j\right)\right)$ obtained by integrating out $\tilde{Z}_l$ from F_2, i.e $F_2^{(1)}(y) \doteq \int F_2(y, z)\mu_l(dz)$, where $y \in \left(H_0^{\otimes l+1} \times \left(\prod_{j=1}^{l-1} \mathcal{M}_j\right)\right)$. Recalling that $||X_l||_0 \le M_0$, a.s., and applying Lemma 3.3 with $E_2 \doteq H_0$, $E_1 \doteq \left(H_0^{\otimes l} \times \left(\prod_{j=1}^{l-1} \mathcal{M}_j\right)\right)$, $K \doteq K_0 \cap \{x \in H_0 : ||x||_0 \le M_0\}$ and $f \doteq F_1 + F_2^{(1)}$, we have that there exists a measurable function $h : \left(H_0^{\otimes l} \times \left(\prod_{j=1}^{l-1} \mathcal{M}_j\right)\right) \to H_0$ satisfying $||h(\cdot)||_0 \le M_0$ such that the right side of (1.12) is bounded below by

$$E^{\tilde{v}}\left(F_1(\mathbf{X}_{l-1}, h(\mathbf{X}_{l-1}, \tilde{\mathbf{Z}}_{l-1})) + F_2^{(1)}(\mathbf{X}_{l-1}, h(\mathbf{X}_{l-1}, \tilde{\mathbf{Z}}_{l-1}), \tilde{\mathbf{Z}}_{l-1})\right) - \epsilon/2(l+1).$$

By subtracting an additional $\epsilon/2(l+1)$ from this lower bound, we can take h to be a continuous map via an application of [8, Theorem V.16a] and the dominated convergence theorem. We now iterate the above procedure l times to obtain the following inequality:

$$E^{\tilde{v}}\left(F_1(\mathbf{X}_l) + F_2(\mathbf{X}_l, \tilde{\mathbf{Z}}_l)\right) \ge E^{\tilde{v}}\left(F_1(\Gamma(\tilde{\mathbf{Z}}_l)) + F_2(\Gamma(\tilde{\mathbf{Z}}_l), \tilde{\mathbf{Z}}_l)\right) - \epsilon,$$

where

- $\Gamma : \prod_{j=1}^{l} \mathcal{M}_j \to H_0^{\otimes l+1}$ is continuous.
- $\Gamma(\mathbf{z}_l)$ can be written $(\Gamma_0, \Gamma_1(\mathbf{z}_1), \dots \Gamma_l(\mathbf{z}_l))$, where $\mathbf{z}_i \doteq (z_1, \dots, z_i) \in \prod_{j=1}^{i} \mathcal{M}_j$,
- Γ_0 is a non-random element of H_0 bounded in norm by M_0,
- for $i = 1, \dots l$ $\Gamma_i : \prod_{j=1}^{i} \mathcal{M}_j \to H_0$ satisfies $\|\Gamma_i(\mathbf{u})\|_0 \le M_0$ for $\mathbf{u} \in \prod_{j=1}^{i} \mathcal{M}_j$.

Now define for $j = 1, \dots, l$ measurable maps Z_j from Ω to $\mathcal{M}_j$ by

$$Z_j(\omega)(s) \doteq W(\omega)(s) - W(\omega)(t_j), \qquad t_j \le s \le t_{j+1}$$

and let $\mathbf{Z}_i \doteq (Z_1, \dots, Z_i)$ for $i \in \{1, \dots l\}$. Finally define

$$\overline{v}(s, \omega) \doteq \Gamma_0 \mathcal{I}_{\{0\}}(s) + \sum_{j=1}^{l} \Gamma_j(\mathbf{Z}_j(\omega)) \mathcal{I}_{(t_j, t_{j+1}]}(s).$$

Clearly $\overline{v}(s)$ is an elementary process in $\mathcal{A}^W$ satisfying $\|\overline{v}(s)\|_0 \le M_0$ for each $s \in [0, T]$ and

$$\begin{aligned} E^{\tilde{v}} & \left(\frac{1}{2} \int_0^T \|v_0(s)\|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} v_0(s) ds\right) \right) \\ \ge \quad & E\left(\frac{1}{2} \int_0^T \|\overline{v}(s)\|_0^2 ds + f\left(W + \int_0^{\cdot} \overline{v}(s) ds\right) \right) - \epsilon. \end{aligned}$$

This proves part 1.
Next, taking $\tilde{v}$ in (1.11) to be a bounded elementary process and $v_0 = \tilde{v}$, we obtain

$$\begin{aligned} & \inf_{\tilde{v} \in \mathcal{A}^{(b)}} E^{\tilde{v}} \left(\frac{1}{2} \int_0^T \|\tilde{v}(s)\|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} \tilde{v}(s) ds\right) \right) \\ \ge \quad & \inf_{v \in \mathcal{A}^{W,(b)}} E\left(\frac{1}{2} \int_0^T \|v(s)\|_0^2 ds + f\left(W + \int_0^{\cdot} v(s) ds\right) \right), \end{aligned} \tag{1.13}$$

where $\mathcal{A}^{W,(b)}$ is the subclass of $\mathcal{A}^W$ of bounded elementary processes. Since elements of $\mathcal{A}^{W,(b)}$ are piecewise constant, for every $v \in \mathcal{A}^{W,(b)}$ we can recursively construct $\tilde{v} \in \mathcal{A}^{(b)}$ such that

$$\begin{aligned} E^{\tilde{v}} & \left(\frac{1}{2} \int_0^T \|\tilde{v}(s)\|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} \tilde{v}(s) ds\right) \right) \\ = \quad & E\left(\frac{1}{2} \int_0^T \|v(s)\|_0^2 ds + f\left(W + \int_0^{\cdot} v(s) ds\right) \right) \end{aligned} \tag{1.14}$$

(see [2, Theorem 3.1] for details). Combining (1.13), (1.14), we have that

$$\begin{aligned}&\inf_{\tilde{v}\in\mathcal{A}^{(b)}} E^{\tilde{v}}\left(\frac{1}{2}\int_0^T \|\tilde{v}(s)\|_0^2 ds + f\left(\tilde{W}+\int_0^{\cdot}\tilde{v}(s)ds\right)\right)\\ &= \inf_{v\in\mathcal{A}^{W,(b)}} E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + f\left(W+\int_0^{\cdot}v(s)ds\right)\right). \quad (1.15)\end{aligned}$$

Next, taking $\tilde{v}\equiv 0$ in (1.11) and observing that $\mathcal{A}^{W,(b)}\subset\mathcal{A}^{(b)}$, it follows

$$\begin{aligned}&\inf_{v\in\mathcal{A}^{W,(b)}} E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + f\left(W+\int_0^{\cdot}v(s)ds\right)\right)\\ &= \inf_{v\in\mathcal{A}^{(b)}} E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + f\left(W+\int_0^{\cdot}v(s)ds\right)\right). \quad (1.16)\end{aligned}$$

Now let $v\in\mathcal{A}$ be such that $E\{\int_0^T \|v(s)\|_0^2 ds\}<\infty$. Choose a sequence $\{v^n : n\in I\!N_0\}$ in $\mathcal{A}$ such that each v^n is a bounded elementary process, $\lim_{n\to\infty} E\left(\int_0^T \|v^n(s)-v(s)\|_0^2 ds\right)=0$, and $\sup_{n\in I\!N_0} E\left(\int_0^T \|v^n(s)\|_0^2 ds\right)<1+E\left(\int_0^T \|v(s)\|_0^2 ds\right)$. Clearly $\int_0^t (v^n(s)-v(s))ds$ converges to zero in probability for each $t\in[0,T]$. Also, an application of Lemma 3.1 shows that $\{\int_0^{\cdot}(v^n(s)-v(s))ds\}$ is tight in $\mathcal{C}([0,T]:H)$. Thus $(W,\int_0^{\cdot} v^n(s)ds)$ converges weakly to $(W,\int_0^{\cdot} v(s)ds)$, and therefore

$$\begin{aligned}&\lim_{n\to\infty} E\left(\frac{1}{2}\int_0^T \|v^n(s)\|_0^2 ds + f\left(W+\int_0^{\cdot}v^n(s)ds\right)\right)\\ &= E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + f\left(W+\int_0^{\cdot}v(s)ds\right)\right).\end{aligned}$$

Using $\mathcal{A}\subset\mathcal{A}^{(b)}$, this proves that

$$\begin{aligned}&\inf_{v\in\mathcal{A}^{(b)}} E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + f\left(W+\int_0^{\cdot}v(s)ds\right)\right)\\ &= \inf_{v\in\mathcal{A}} E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + f\left(W+\int_0^{\cdot}v(s)ds\right)\right). \quad (1.17)\end{aligned}$$

The proof of part 2 is completed by combining (1.15), (1.16) and (1.17). ■

We now present the main result of this section. Though in the theorem we take f to be a bounded function, it can be shown (as in [2]) that the representation continues to hold if f is bounded from above.

Theorem 3.6 *Let f be a bounded, Borel measurable function mapping $\mathcal{C}([0,T]:H)$ into $I\!R$. Then*

$$-\log E\exp\{-f(W)\} = \inf_{v\in\mathcal{A}} E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + f\left(W + \int_0^{\cdot} v(s)ds\right)\right) \tag{1.18}$$

Proof: The proof presented here is adapted from that of the finite dimensional case studied in [2]. We claim that it suffices to prove the result for f that are continuous. To see this, let $\{f^{(n)}\}$ be a sequence of real valued continuous functions on $\mathcal{C}([0,T],H)$ such that $\sup_{x,n}|f^{(n)}(x)| \le \sup_x |f(x)|$ and $f^{(n)}$ converges to f $\theta-$a.s. An application of dominated convergence theorem shows that

$$-\log E\exp\{-f^{(n)}(W)\} \to -\log E\exp\{-f(W)\}.$$

For $\mathcal{B}\subset\mathcal{A}$ and g a bounded, Borel measurable function mapping $\mathcal{C}([0,T]:H)$ into $I\!R$ define

$$\Lambda(\mathcal{B},g) \doteq \inf_{v\in\mathcal{B}} E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + g\left(W + \int_0^{\cdot} v(s)ds\right)\right).$$

To prove the claim, we must show that $\Lambda(\mathcal{A},f^{(n)})$ converges to $\Lambda(\mathcal{A},f)$ as $n\to\infty$. Let $K \doteq \sup_x |f(x)|$ and $\overline{\mathcal{A}} \doteq \{v\in\mathcal{A} : E\left(\int_0^T \|v(s)\|_0^2 ds\right) \le 4K\}$. Then clearly $\Lambda(\mathcal{A},f^{(n)})$ equals $\Lambda(\overline{\mathcal{A}},f^{(n)})$ and $\Lambda(\mathcal{A},f)$ equals $\Lambda(\overline{\mathcal{A}},f)$. Let $\epsilon > 0$ be arbitrary. Choose $N\in I\!N_0$ such that $\frac{4K^2}{N} \le \epsilon/2$. Fix $v\in\overline{\mathcal{A}}$ and define the stopping time $\tau_N \doteq \inf\{s\in[0,T] : \int_0^s \|v(s)\|_0^2 \ge N\}\wedge T$. Recall that $\mathcal{A}_N \doteq \{v\in\mathcal{A} : \int_0^T \|v(s)\|_0^2 ds \le N, \theta - a.s.\}$. Let $v_N\in\mathcal{A}_N$ be defined as $v_N(s) \doteq v(s)\mathcal{I}_{[0,\tau_N]}(s)$, where $\mathcal{I}$ denotes the indicator function. We observe that

$$\begin{aligned}\Lambda(\mathcal{A}_N,f^{(n)}) &\le E\left(\frac{1}{2}\int_0^T \|v_N(s)\|_0^2 ds + f^{(n)}\left(W + \int_0^{\cdot} v_N(s)ds\right)\right)\\ &\le E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + f^{(n)}\left(W + \int_0^{\cdot} v(s)ds\right)\right) + \epsilon,\end{aligned}$$

where the second line in the display above follows since $v\in\overline{\mathcal{A}}$ implies that the probability of the set $\{\tau_N < T\}$ is at most $\frac{4K}{N}$. Taking the infimum over all $v\in\overline{\mathcal{A}}$ in the inequality above we have that

$$\Lambda(\mathcal{A},f^{(n)}) \le \Lambda(\mathcal{A}_N,f^{(n)}) \le \Lambda(\mathcal{A},f^{(n)}) + \epsilon.$$

Exactly the same argument gives

$$\Lambda(\mathcal{A},f) \le \Lambda(\mathcal{A}_N,f) \le \Lambda(\mathcal{A},f) + \epsilon.$$

Finally, an application of Lemma 3.4 shows that $\Lambda(\mathcal{A}_N, f^{(n)})$ converges to $\Lambda(\mathcal{A}_N, f)$ as $n \to \infty$. This proves the claim.
Henceforth we will assume that f is continuous. We will prove that the left side of (1.18) is bounded above and below by the right side.

Proof of the upper bound: From Proposition 1.4.2 of [10] it follows that

$$-\log E \exp\{-f(W)\} = \inf_{\gamma\in\mathcal{P}(\Omega):\gamma\ll\theta} \{R(\gamma\|\theta) + E^{\gamma}(f(W))\}\,, \qquad (1.19)$$

where $\mathcal{P}(\Omega)$ is the class of all probability measures on $(\Omega, \mathcal{F})$. Let $\tilde{v} \in \mathcal{A}$ be a bounded elementary process. Clearly, $(\int_0^t \langle\tilde{v}(s), dW(s)\rangle, \mathcal{F}_t)_{0\le t\le T}$ is a real valued continuous martingale with quadratic variation $\int_0^t \|\tilde{v}(s)\|_0^2 ds$ ([5, Section 4]). The boundedness assumption also implies that the expectation $E(\exp\{\int_0^T \|\tilde{v}(s)\|_0^2 ds\})$ is finite, and therefore Proposition 5.12 of [18] yields

$$E\left(\exp\left\{\int_0^T \langle\tilde{v}(s), dW(s)\rangle - \frac{1}{2}\int_0^T \|\tilde{v}(s)\|_0^2 ds\right\}\right) = 1.$$

By Theorem 2.1 $d\gamma^{\tilde{v}} \doteq \exp\left\{\int_0^T \langle\tilde{v}(s), dW(s)\rangle - \frac{1}{2}\int_0^T \|\tilde{v}(s)\|_0^2 ds\right\} d\theta$ is a probability measure and, under $\gamma^{\tilde{v}}$, $\tilde{W}(t) \doteq W(t) - \int_0^t \tilde{v}(s)ds$ is a $Q-$Wiener process. Next from the definition of the relative entropy function

$$R(\gamma^{\tilde{v}}\|\theta) \doteq E^{\tilde{v}}\left(\log\frac{d\gamma^{\tilde{v}}}{d\theta}\right) = E^{\tilde{v}}\left(\int_0^T \langle\tilde{v}(s), dW(s)\rangle - \frac{1}{2}\int_0^T \|\tilde{v}(s)\|_0^2 ds\right), \qquad (1.20)$$

where $E^{\tilde{v}}$ denotes the expectation with respect to $\gamma^{\tilde{v}}$. The expression on the right side above equals $\frac{1}{2}E^{\tilde{v}}\left(\int_0^T \|\tilde{v}(s)\|_0^2 ds\right)$, and consequently

$$R(\gamma^{\tilde{v}}\|\theta) + E^{\tilde{v}}(f(W)) = E^{\tilde{v}}\left(\frac{1}{2}\int_0^T \|\tilde{v}(s)\|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} \tilde{v}(s)ds\right)\right).$$

It follows now from (1.19) that

$$-\log E\exp\{-f(W)\} \le E^{\tilde{v}}\left(\frac{1}{2}\int_0^T \|\tilde{v}(s)\|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} \tilde{v}(s)ds\right)\right) \qquad (1.21)$$

for every $\tilde{v}$ as above. Therefore, an application of part 2 of Lemma 3.5 yields

$$\begin{aligned} & -\log E\exp\{-f(W)\} \\ \le\ & \inf_{\tilde{v}\in\mathcal{A}^{(b)}} E^{\tilde{v}}\left(\frac{1}{2}\int_0^T \|\tilde{v}(s)\|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} \tilde{v}(s)ds\right)\right) \\ =\ & \inf_{v\in\mathcal{A}} E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + f\left(W + \int_0^{\cdot} v(s)ds\right)\right). \end{aligned}$$

Proof of the lower bound: From Proposition 1.4.2 of [10] we have that

$$-\log E\exp\{-f(W)\} = R(\gamma_0\|\theta) + E^{\gamma_0}(f(W)), \tag{1.22}$$

where $\frac{d\gamma_0}{d\theta} \doteq c\exp\{-f(W)\}$ a.s. and c is the normalizing constant. Define $L(t) = E(\frac{d\gamma_0}{d\theta}|\mathcal{G}_t)$. Clearly $(L(t),\mathcal{G}_t)_{0\le t\le T}$ is a right continuous martingale bounded above and below by $e^{2\|f\|_\infty}$ and $e^{-2\|f\|_\infty}$ respectively. It follows from Theorem 2.2 that there exists $u\in\mathcal{A}^W$ such that for all $0\le t\le T$

$$L(t) = 1 + \int_0^t \langle u(s), dW(s)\rangle.$$

We can rewrite the last equality as

$$L(t) = 1 + \int_0^t \langle \tilde{v}(s)L(s), dW(s)\rangle,$$

where $\tilde{v}(t) \doteq u(t)/L(t)$. Since $L(t)$ is a real valued continuous nonnegative martingale with $L(0)\equiv 1$, we have ([15, Lemma 7.1.4]) that

$$L(t) = \exp\left(\int_0^t \langle \tilde{v}(s), dW(s)\rangle - \frac{1}{2}\int_0^t \|\tilde{v}(s)\|_0^2 ds\right).$$

It follows from Theorem 2.1 that under γ_0

$$\tilde{W} \doteq W - \int_0^{\cdot} \tilde{v}(s)ds$$

is a Brownian motion with covariance Q. Therefore

$$-\log E\exp\{-f(W)\} = E^{\tilde{v}}\left(\frac{1}{2}\int_0^T \|\tilde{v}(s)\|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} \tilde{v}(s)ds\right)\right). \tag{1.23}$$

As in the proof of part 2 of Lemma 3.5 we can approximate $\tilde{v}$ by a sequence $\{\tilde{v}^n, n\in \mathbb{N}_0\}$ of bounded elementary processes in $\mathcal{A}$ such that $E^{\tilde{v}}\left(\int_0^T \|\tilde{v}^n(s)\|_0^2 ds\right) \le 1 + E^{\tilde{v}}\left(\int_0^T \|\tilde{v}(s)\|_0^2 ds\right)$ and $E^{\tilde{v}}\left(\int_0^T \|\tilde{v}^n(s) - \tilde{v}(s)\|_0^2 ds\right) \to 0$ as $n\to\infty$. It now follows, as in the proof of part 2 of Lemma 3.5, that

$$E^{\tilde{v}}\left(\frac{1}{2}\int_0^T \|\tilde{v}^n(s)\|_0^2 ds + f(\tilde{W} + \int_0^{\cdot} \tilde{v}^n(s)ds)\right)$$

converges to the right side of (1.23) as $n\to\infty$.

Let $\epsilon > 0$ be arbitrary. We have shown that there exists $M_0 < \infty$ and an elementary process $v_0\in\mathcal{A}$ satisfying $||v_0(s)||_0 \le M_0$ for all $s\in[0,T]$ such that

$$-\log E\exp\{-f(W)\} \ge E^{\tilde{v}}\left(\frac{1}{2}\int_0^T \|v_0(s)\|_0^2 ds + f\left(\tilde{W} + \int_0^{\cdot} v_0(s)ds\right)\right) - \epsilon.$$

The proof is now completed by applying part 1 of Lemma 3.5. ■

4 Hilbert space valued diffusions

In this section we will state an LDP for Hilbert space valued diffusions. Due to space limitations many details have been omitted. We refer the interested reader to [3] for complete proofs. Let H and K be two separable Hilbert spaces. As before, let W be a $H-$valued Brownian motion with the trace class covariance Q. We are interested in diffusion processes that take values in K. Let $\mathcal{L}(H,K)$ be the space of all bounded linear operators from H to K and denote the operator norm for $A \in \mathcal{L}(H,K)$ by $\|A\|_{op}$. We begin with a result proved in [19] concerning existence and strong uniqueness of solutions to $K-$valued stochastic differential equations. Let $A : K \to K$ and $G : K \to \mathcal{L}(H,K)$ be measurable mappings. Throughout this section the following conditions will be assumed on these functions. No other conditions on the stochastic differential equation will be required.

1. For every $n \in I\!N_0$ there exists $L_n < \infty$ such that for all $x, y \in K$ satisfying $\|x\| \le n$ and $\|y\| \le n$,
$$\|A(x) - A(y)\| + \|G(x) - G(y)\|_{op} \le L_n \|x - y\|. \qquad (1.24)$$
2. There exists $B < \infty$ such that for all $x \in K$
$$\|G(x)\|_{op}^2 \le B(1 + \|x\|^2) \text{ and } \langle x, A(x)\rangle \le B(1 + \|x\|^2). \qquad (1.25)$$

Theorem 4.1 *The stochastic differential equation:*

$$dX(t) = A(X(t))dt + G(X(t))dW(t), \ X(0) = x \qquad (1.26)$$

has a unique strong solution as a $K-$valued process that satisfies

$$\int_0^T \|G(X(s))\|_{op}^2 \, ds < \infty \ \textit{and} \ \int_0^T \|A(X(s))\| ds < \infty \ \textit{a.s.}$$

Although [19] explicitly considers only the case $H = K$, the proof applies to the the more general setup of the theorem above with only notational changes.

The following proposition is a consequence of Theorems 4.1. and 2.1.

Proposition 4.2 *Let $v \in A$. Then there exists a unique $K-$valued progressively measurable process $\{X^v(t); 0 \le t \le T\}$ such that*

1.
$$\theta\left\{\int_0^T \|G(X^v(s))\|_{op}^2 \, ds + \int_0^T \|A(X^v(s))\| ds < \infty\right\} = 1,$$

2. for every $t \in [0,T]$

$$X^v(t) = x + \int_0^t G(X^v(s))v(s)ds + \int_0^t A(X^v(s))ds + \int_0^t G(X^v(s))dW(s). \tag{1.27}$$

Proof: Consider first the case when $v \in \mathcal{A}$ is such that

$$E\left(\exp\left\{-\int_0^T \langle v(s), dW(s)\rangle - \frac{1}{2}\int_0^T \|v(s)\|_0^2 ds\right\}\right) = 1.$$

In this case from Theorem 2.1 $\tilde{W} \doteq W + \int_0^\cdot v(s)ds$ is a $Q-$Wiener process under the probability measure

$$d\gamma^v \doteq \exp\left\{-\int_0^T \langle v(s), dW(s)\rangle - \frac{1}{2}\int_0^T \|v(s)\|_0^2 ds\right\} d\theta.$$

Let X^v be the unique solution to (1.26) with W replaced by $\tilde{W}$ on $(\Omega, \mathcal{F}, \gamma^v)$. Clearly X^v solves (1.27) a.s. θ and conditions 1 and 2 in the proposition are automatically satisfied. This shows existence. Uniqueness is argued in a similar manner using change of measure techniques. Now let $v \in \mathcal{A}$ be arbitrary and define $\tau_n \doteq \inf\{s \in [0,T] : \int_0^s \|v(s)\|_0^2 > n\} \wedge T$, where the infimum is defined to be ∞ if the set over which the infimum is taken is empty. Clearly τ_n increases to T, and in fact for every $\omega \in \Omega$ outside a θ-null set there exists a finite number $M(\omega)$ such that $\tau_n(\omega) \equiv T$ for all $n \geq M(\omega)$. Define $v^n(s) \doteq v(s)\mathcal{I}_{[0,\tau_n]}(s)$ for $0 \leq s \leq T$, where $\mathcal{I}_{[a,b]}(s)$ is the indicator function of the interval $[a,b]$. From the case considered previously we know there exist processes $\{X^{n,v}(t)\}_{0\leq t\leq T}$ that satisfy (1.27) with v replace by v^n. Also, by the strong uniqueness, $X^{m,v}(t) = X^{n,v}(t)$ a.s. for all $t \leq \tau_n$ and $m \geq n$. Define $X^v(t) \doteq X^{n,v}(t)$ for $t \leq \tau_n$. Clearly $X^v(t)$ is defined for $0 \leq t \leq T$ a.s. and by its construction and strong uniqueness solves (1.27). We leave it to the reader to verify that condition 2 is satisfied by X^v. For uniqueness, it is enough to note that if X^v and Y^v solve (1.27), then a.s. $X^v(t) = Y^v(t)$ for $t \leq \tau_n$ for every $n \in \mathbb{N}_0$. ■

The next proposition follows from the representation obtained in Theorem 3.6 and the fact that since X is a strong solution to (1.26) there exists a Borel measurable map $h : \mathcal{C}([0,T] : H) \to \mathcal{C}([0,T] : K)$ such that $X(t) = \pi(t) \circ h(W)\ \theta$ - a.s., where $\pi(t)$ is the coordinate mapping on $\mathcal{C}([0,T] : K)$.

Proposition 4.3 *Let* A *and* G *be as in Theorem 4.1 and let* X *be the unique solution to (1.26). Then for any bounded Borel function* $f : \mathcal{C}([0,T] : H) \to R$

$$-\log E\exp\{-f(X)\} = \inf_{v\in\mathcal{A}} E\left(\frac{1}{2}\int_0^T \|v(s)\|_0^2 ds + f(X^v)\right),$$

where X^v *is the unique solution to (1.27).*

The following is the main result of this section. Define

$$\mathcal{D}_f \doteq \left\{ v \in L^2([0,T] : H_0) : f(t) = x + \int_0^t A(f(s))ds + \int_0^t G(f(s))v(s)ds \right\}.$$

Theorem 4.4 *Let G and A be as in Theorem 4.1. Let $\{X^\epsilon\}$ solve the equation*

$$dX^\epsilon(t) = A(X^\epsilon(s))ds + \sqrt{\epsilon}G(X^\epsilon(s))dW(s), \ X^\epsilon(0) = x. \tag{1.28}$$

Then $\{X^\epsilon\}$ satisfies the Laplace principle in $\mathcal{C}([0,T] : K)$ with rate function

$$I_x(f) \doteq \inf_{v \in \mathcal{D}_f} \left\{ \frac{1}{2} \int_0^T \|v(t)\|_0^2 dt \right\},$$

where the infimum over the empty set is taken to be ∞.

As noted previously, space limitations prohibit giving the proof of this theorem. However, it is worth noting that given the representation in Proposition 4.3 and the tightness and convergence results in Lemmas 3.1 and 3.2, the proof is very similar to the finite dimensional case in [2]. In fact, the only significant difference is an additional localization argument needed to deal with the fact that A and G can be unbounded as functions on H. For full details, we refer to [3].

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