STAT 155 Introductory Statistics

Lecture 11: Randomness and Probability Model
The Monty Hall Problem

• *Let’s Make A Deal:* a game show back in the 90’s.

• A player is given the choice of three doors. Behind one door is the Grand Prize (a car and a cruise); behind the other two doors, booby prizes (stinking pigs). The player picks a door, and the host peeks behind the doors and opens one of the rest of the doors. There is a booby prize behind the open door.

• The host offers the player either to stay with the door that was chosen at the beginning, or to switch to the remaining closed door.

• Which is better: to switch doors or to stay with the original choice? What are the chances of winning in either case?
Three prisoners, A, B, and C are on death row. The governor decides to pardon one of the three and chooses at random the prisoner to pardon.

He informs the warden of his choice but requests that the name be kept secret for a few days.

The next day, A tries to get the warden to tell him who had been pardoned.

The warden refuses. A then asks which of B or C will be executed. The warden thinks for a while, then tells A that B is to be executed.

Can A increase his chance of survival by swapping his fate with C?
The previous two problems are equivalent.

Play it online at

How can we solve similar problems systematically?
- Probability models.
Randomness & Probability

• We call a phenomenon (or an experiment) random if individual outcomes are uncertain, but a regular distribution of outcomes emerges with a large number of repetitions.
  – Example: Toss a coin, gender of new born baby.

• The probability of any outcome in a random experiment is the proportion of times the outcome would occur in a very long series of independent repetitions,
  – i.e., probability is long-term relative frequency.
  – In the early days, probability was associated with games of chance (gambling).
Probability as long term relative frequency
Probability Model

• Probability models attempt to model random behavior.
• Consist of two parts:
  – A list of possible outcomes (sample space $S$)
  – An assignment of probabilities $P$ to each outcome
• The probability of an event $A$, denoted by $P(A)$, can be considered as the long run relative frequency of the event $A$. 
Sample Space and Events

• **Sample space** $S$: the set of all possible outcomes in a random experiment.
  – Examples: Toss a coin. Record the side facing up.
    $$S = \{\{\text{Heads}\}, \{\text{Tails}\}\} = \{\text{H}, \text{T}\}.$$  
  – Toss a coin twice. Record the side facing up each time.
    $$S = \text{?}.$$  
  – Toss a coin twice. Record the number of heads in the two tosses. $S = \text{?}.$

• **Event**: An outcome or a set of outcomes in a random experiment,
  – i.e. a subset of the sample space.
Sample Space
a sample space of a random experiment is the set of all possible outcomes.

Simple events
The individual outcomes are called simple events.

Event
An event is a collection of one or more simple events.

Our objective is to determine \( P(A) \), the probability that an event \( A \) will occur.
Sample space $S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$.

- There are 8 simple events, among which are
  
  $E_1 = \{\text{HHH}\}$ and $E_8 = \{\text{TTT}\}$.

- Some compound events include
  
  $A = \{\text{at least two heads}\} = \{\text{HHH, HHT, HTH, THH}\}$.
  
  $B = \{\text{exactly two tails}\} = ?$.  

Toss a coin 3 times
Boy or girl?

- An experiment in a hospital consists of recording the gender of each newborn infant until the birth of a male is observed.
- The sample space of this experiment is 
  \[ S = \{M, FM, FFM, FFFM, \ldots\} \]
- The sample space contains an infinite number of outcomes.
Release from the death-row

• An executioner offers a death-row prisoner a final chance to gain his release.
• 20 chips, 10 white and 10 blue, will be put into two urns by the prisoner with each contains at least one chip.
• The executioner will pick one urn randomly and from that urn, one chip at random.
• If the chip is white, the prisoner will be set free; if it is blue, he will be executed.
• Sample Space: $S = \{(1,0), (9,10)\}, ((0,1), (10, 9))$, $(2, 0), (8, 10)], \ldots, [(9,10), (1, 0)]$} (count carefully!)
• What’s the best allocation for the prisoner? (intuition?)
Basic Concepts

- The *complement* of an event $A$
  - the set of all outcomes in $S$ that are not in $A$.
  - $\{ \text{not } A \}$
- The *union* of two events $A$ and $B$
  - the event consisting of all outcomes that are *either* in $A$ *or* in $B$ *or* in both.
  - $A \cup B$
- The *intersection* of two events $A$ and $B$
  - the event consisting of all outcomes that are in both events.
  - $A \cap B$
- When two events $A$ and $B$ have no outcomes in common, they are said to be *disjoint (or mutually exclusive)* events.
Venn Diagram
Example

- The experiment: toss a coin 10 times and the number of heads is recorded.
- Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{1, 3, 5, 7, 9\}$, $C = \{0, 1, 2, 3, 4, 5\}$.
- $S = ?$
- $A \cup B = ?$
- $A \cap B = ?$
- $\{not \ C\} = ?$
- $A \cap C = ?$
Probability Rules

• For any event $A$, $0 \leq P(A) \leq 1$.

• $P(S) = 1$.

• If $A$ and $B$ are disjoint events, then
  $$P(A \cup B) = P(A) + P(B).$$ (addition rule for disjoint events)

• For any event $A$, $P(\text{not } A) = 1 - P(A)$. (complement rule)

• For any two events $A$ and $B$,
  $$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$ (general addition rule)

• If $A$ and $B$ are disjoint, then $P(A \cap B) = 0$. 
Equally Likely Outcomes

• If there are \( k \) equally likely outcomes, then the probability assigned to each outcome is \( \frac{1}{k} \).

\[
P(A) = \frac{\text{(# of outcomes in } A)}{k}
\]

• Key: smart counting
  --- ``no omission, no duplication’’
Roll a fair die once

- The label facing up, when a fair die is rolled, is observed.
- Sample Space: \( S = \{1, 2, 3, 4, 5, 6\} \).
- Every outcome is equally likely to occur.
  \[
  P(1) = P(2) = \ldots = P(6) = \frac{1}{6}.
  \]
Roll a fair die once

• Consider the following events
  – \( A \): The label observed is at most 2.
  – \( B \): The label observed is an even number.
  – \( C \): Label 4 turns up.

• Find
  
  • \( P(A) \)
  • \( P(\text{not } A) \)
  • \( P(A \text{ and } B) \)
  • \( P(A \text{ or } C) \)
  • \( P(A \text{ or } B) \)
Cards

A card is drawn from an ordinary deck of 52 playing cards. What is the probability that the card is
-- a club?
-- a king?
-- a club and a king?
-- a club or a king?
-- neither a club nor a king?
Glasses

- In a group of 88 people in Stat 155, 11 out of 50 women and 8 out of 38 men wear glasses.
- What is the probability that a person chosen at random from the group is a woman or someone who wears glasses?
Venn diagram with 3 events

- $A = \{\text{Google stock moves up today}\}$
- $B = \{\text{Walmart stock moves up today}\}$
- $C = \{\text{Exxon stock moves up today}\}$

$P(A) = 0.1$, $P(B) = 0.2$, $P(C) = 0.5$

$P(A \cap B) = 0.05$, $P(A \cap C) = 0.04$, $P(B \cap C) = 0.02$

$P(A \cap B \cap C) = 0.01$

Find:

(i) $P(\text{at least one of the 3 stocks go up}) =$

(ii) $P(\text{both Google and Exxon go down}) =$

(iii) $P(\text{only one of the 3 sticks goes up}) =$
How to complete a Venn diagram?

--- Insert a probability in each disjoint part

--- "inside-out"

--- See details on the board …
Take Home Message

- sample space, outcome, event
- union (or), intersection (and), complement (not), disjoint
- Venn diagram
- Basic rules:
  - For any event $A$, $P(\text{not } A) = 1 - P(A)$.
  - If $A$ and $B$ are disjoint, then $P(A \cap B) = 0$.
  - For any two events $A$ and $B$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. 