Lecture 10 Brownian Motion

Suppose in a discrete-time market the returns of a stock price process $\{S(t)\}$ is modeled by the stochastic difference equation

$$\frac{S(t+\delta)-S(t)}{S(t)} = \mu\ \delta + \sigma\sqrt{\delta}\ \epsilon_t \tag{10.1}$$

with a time increment δ, a drift parameter μ, a volatility parameter σ, and a noise random variable (innovation) $\epsilon_t \sim N(0,1)$ which is independent of $\mathcal{F}_t$ (the history up to time t). Heuristically, letting $\delta \to 0$ will turn (10.1) to a stochastic differential equation (still abbreviated to SDE)

$$dS_t/S_t = \mu\ dt + \sigma\ dW_t, \tag{10.2}$$

which has an explicit solution

$$S_t = S_0\ \exp[(\mu - \sigma^2/2)\ t + \sigma W_t]. \tag{10.3}$$

Can we verify the solution by differentiating (10.3)? If you do it superficially by using the Newton-Leibniz calculus, you would not get back to (10.2). The reason is that $\{W_t\}$ is a very special stochastic process, arguably the most important one in probability theory, named *Brownian motion.*

Louis Bachelier, a Ph.D. student of Henri Poincaré, introduced Brwonian motion in 1900 as a model for the dynamic behavior of the Paris stock market. Notice that it took place 5 years before Albert Einstein developed a physical model of Brownian motion to describe small particles suspended in a liquid, and 23 years before Norbert Wiener gave the first rigorous mathematical construction of Brownian motion. For that reason, Bachelier is now considered by many as the founder of modern mathematical finance. See the article by Robert Jarrow and Philip Protter for the historical summary.

In what follows, we will state a number of important facts regarding Brownian motion. All proofs are skipped. See the book *Brownian Motion and Stochastic Calculus* (1991, Springer) by Karatzas and Shreve for details.

10.1 Definition of Brownian motion and a limit of random walks

Let $\mathbb{R}_+$ denote the half straight line, i.e. the set of all nonnegative real numbers, which serves as an index set in continuous-time finance. Sometimes we write $t \geq 0$ for $t \in \mathbb{R}_+$.

Definition 10.1 *A standard Brownian motion (or standard Wiener process) is a stochastic process* $W \triangleq \{W_t\}_{t\geq 0}$, *i.e. a collection of random variables* W_t *defined on the same probability space* $(\Omega, \mathcal{F}, P)$, *satisfying the following conditions:*

[1] $W_0 = 0$;

[2] *with probability one, the function W_t is continuous in t;*

[3] *W has stationary and independent increments, i.e. for any positive integer n and any $0 = t_0 < t_1 < \cdots < t_n$, the random variables $W_{t_i} - W_{t_{i-1}}$, $i = 1, ..., n$ are mutually independent, and $W_{s+t} - W_s$ has the same distribution as W_t for any $s, t > 0$;*

[4] $W_t \sim N(0, t)$.

It is not obvious that such a process W exists. In particular, why should [3]–[4] be compatible with [2] (continuous sample paths)? Wiener (1923) was the first to rigorously show the existence of Brownian motion.

The close connection between Brownian motion and random walks is one of the most important facts in probability theory. Let $\xi_1, \xi_2, ...$ be iid random variables with mean 0 and variance 1. For positive integer n, define a continuous-time process $W^{(n)} \triangleq \{W_t^{(n)}\}_{t\geq 0}$ by

$$W_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \xi_i, \tag{10.4}$$

which is a random step function with jumps of size $\xi_i/\sqrt{n}$ at time i/n, $i = 1, ..., [nt]$, where $[x]$ denotes the integer part of $x \in I\!R$. Although intuitively reasonable, it takes some hard technical steps to justify that as n goes to infinity, the distribution of random function $W^{(n)}$ approaches (in a certain sense) to that of W. See Billingsley's book on "weak convergence" for details. Furthermore, many functions of $W^{(n)}$ would approximate the corresponding functions of W, in distribution or in expectation, etc. Results of this kind are under the name "invariance principle". If we believe this, then the Black-Scholes option pricing formula in Section 4.3 would be a special case of such a principle. Moreover, the maximum function $\max_{0\leq u\leq t} W_u^{(n)}$ converges in distribution to $\max_{0\leq u\leq t} W_u$, and the first passage time $\tau^{(n)}(a) = \inf\{t \geq 0 : W_t^{(n)} \geq a\}$ $(a > 0)$ converges in distribution to $\tau(a) = \inf\{t \geq 0 : W_t = a\}$, etc.

Exercise 10.1 Show that each of the following processes is also a standard Brownian motion: (i) $\{-W_t\}_{t\geq 0}$ (reflection); (ii) $\{W_{s+t} - W_s\}_{t\geq 0}$ (translation), where $s > 0$ is fixed; (iii) $\{aW_{t/a^2}\}_{t\geq 0}$ (scaling), where $a > 0$ is fixed; (iv) $\{tW_{1/t}\}_{t>0}$ (Can you show that $tW_{1/t} \longrightarrow 0$ as $t \downarrow 0$ with probability one?).

10.2 Brownian motion as a Markov process

Brownian motion enjoys the Markov property, the strong Markov property, and its transition density function, called the Gaussian kernel, satisfies a certain partial differential equation (PDE).

First, the collection $\{\mathcal{F}_t\}_{t\in R_+}$ is called the Brownian filtration, where $\mathcal{F}_t$ is the σ-algebra containing all information about W up to time t. Roughly speaking, the Markov property states that for any $0 \le t < s$, the conditional density of W_s given $\mathcal{F}_t$ is the same as the conditional density of W_s given W_t.

Second, a nonnegative random variable τ is called a stopping time with respect to the Brownian filtration if for every $t > 0$, the event $\{\tau \le t\}$ is an element of $\mathcal{F}_t$. Associated with a stopping time τ is a σ-algebra $\mathcal{F}_\tau$, containing all information about W up to τ.

Third, Brownian motion W also satisfies the strong Markov property — a useful generalization of the Markov property:

Theorem 10.1 *Let τ be a stopping time with respect to the Brownian filtration $\{\mathcal{F}_t\}_{t\in R_+}$. For $t \ge 0$, define*

$$W_t^* = W_{\tau+t} - W_\tau,$$

and let $\{\mathcal{F}_t^*\}_{t\in R_+}$ be the filtration of the process $W^* \triangleq \{W_t^*\}_{t\ge 0}$. Then W^* is also a standard Brownian motion; and for any $t > 0$, $\mathcal{F}_t^*$ is independent of $\mathcal{F}_\tau$.

This theorem implies that Brownian motion begins anew at τ and forgets its pre-τ history.

Fourth, it can be verified straightforwardly that for any $s, t > 0$ and $x, y \in I\!R$, the infinitesimal transition probability has the expression

$$P(W_{s+t} \in dy | W_s = x) = p_t(y|x)\ dy = \frac{1}{\sqrt{2\pi t}}\ \exp[-(y-x)^2/2t]\ dy. \tag{10.5}$$

In fact, the transition density $p_t(y|x)$ is the fundamental solution of the heat equation, which means not only

$$\frac{\partial p_t(y|x)}{\partial t} = \frac{1}{2}\frac{\partial^2 p_t(y|x)}{\partial x^2}, \tag{10.6}$$

a stronger result holds:

Theorem 10.2 *Let $f : I\!R \to I\!R$ be a continuous function with polynomial growth at infinity. Then the unique solution $u_t(x)$ to the initial value problem*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \qquad \text{and} \qquad u_0(x) = f(x) \tag{10.7}$$

is given by

$$u_t(x) = Ef(W_t + x) = \int_{-\infty}^{\infty} p_t(y|x) f(y)\ dy. \tag{10.8}$$

The link between Brownian motion and the heat equation makes it possible to study various problems by using either a probabilistic approach or an analytic approach. One such an example is that the Black-Scholes pricing formula can be expressed as a conditional expectation under a risk neutral measure Q or as a solution to some parabolic PDE with a certain initial condition.

10.3 Brownian motion as a Gaussian process and a martingale

$X \triangleq \{X_t,\ t \geq 0\}$ is called a Gaussian process if for any positive integer k and any $0 \leq t_1 < t_2 < \cdots < t_k$, the joint distribution of random vector $(X_{t_1}, ..., X_{t_k})$ is a k-variate Gaussian. The probability distribution of a Gaussian process X is determined by its mean function $m(t) = EX_t$, $t \geq 0$ and covariance function $\sigma(s,t) = Cov(X_s, X_t),\ s,t \geq 0$. Clearly, Brownian motion W is a Gaussian process with $m(t) \equiv 0$ and $\sigma(s,t) = min\{s,t\}$.

$X \triangleq \{X_t,\ t \geq 0\}$ is called a continuous-time martingale with respect to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if for any $0 \leq t < s$, $E(X_s|\mathcal{F}_t) = X_t$ with probability one. It can be verified that Brownian motion W is a martingale, so is the process $\{W_t^2 - t\}_{t\geq 0}$, both with respect to the Brownian filtration. The martingale property of Brownian motion will be used frequently later in proving many important results, especially in the development of Itô's stochastic integral.