

Lecture 14 Introduction to Black-Scholes Theory

The celebrated Black-Scholes (BS) theory in financial economics lays a foundation for three important aspects: a probability model, an option pricing formula and a statistical inference procedure for volatilities. The past few decades have seen a great deal of progress in various extensions of the BS theory. This lecture contains an informal introduction to these three elements, which paves a way for more general discussions in Lecture 15.

14.1 BS SDE — the geometric Brownian motion

In the simplest form, the (continuous-time) BS market contains only two underlying assets: a riskless asset B (bank account) with a constant interest rate $r > 0$ such that for $t \in \mathbb{R}_+$,

$$B_t = B_0 e^{rt}, \quad \text{where } B_0 = 1; \quad (14.1)$$

and a risky asset S (stock) that follows the SDE

$$dS_t/S_t = \mu dt + \sigma dW_t, \quad (14.2)$$

where S_t represents a stock price at time $t \in \mathbb{R}_+$, the constants $\mu \in \mathbb{R}$ and $\sigma > 0$ represent the expected rate of return and volatility respectively, and $\{W_t\}$ is a standard Brownian motion. Recall from Example 13.2 that (14.2) has a solution

$$S_t = S_0 \exp[(\mu - \sigma^2/2)t + \sigma W_t]. \quad (14.3)$$

The general SDE theory (we skip it) assures that (14.3) is the unique solution of (14.2) in some sense. To model a more realistic market, there should be more risky assets, each following its own stochastic dynamics. That would give us more freedom to hedge various kinds of risk.

14.2 BS option pricing formulas

Let $t \leq T$. A contingent claim $g(S_T)$ written on the stock price S_T can be given a time t “fair” price $P(t, S_t)$ in two equivalent ways. The analytical approach yields $P(t, x)$, $t \in [0, T]$, $x \geq 0$ as the solution to the boundary value problem

$$\begin{aligned} \frac{\partial P(t, x)}{\partial t} + rx \frac{\partial P(t, x)}{\partial x} + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 P(t, x)}{\partial x^2} - r P(t, x) &= 0 \\ \text{with } P(T, x) &= g(x). \end{aligned} \quad (14.4)$$

The probabilistic approach expresses $P(t, S_t)$ as a conditional expectation given \mathcal{F}_t under a risk-neutral probability measure Q :

$$P(t, S_t) = E_Q \left[e^{-r(T-t)} g(S_T) \mid \mathcal{F}_t \right], \quad (14.5)$$

where Q is defined through its Radon-Nikodým derivative with respect to the real-world probability measure P (also referred to as the physical measure or objective measure) that governs the BS SDE (14.2):

$$\frac{dQ}{dP} \Big|_{\mathcal{F}_t} = \exp \left[-\frac{\mu - r}{\sigma} W_t - \frac{t}{2} \left(\frac{\mu - r}{\sigma} \right)^2 \right], \quad (14.6)$$

and $\{\mathcal{F}_t, t \geq 0\}$ is the Brownian filtration. Note in this case, the information up to time t , i.e. \mathcal{F}_t , is summarized by the current state S_t . In the special case of an European call option, we have $g(S_T) = \max\{S_T - K, 0\}$, where the positive constant K is a prescribed strike price at the maturity time T . The time t price $P(t, S_t)$, denoted by a function $C^{BS}(S_t, \sigma^2, r, K, T - t)$, enjoys a closed form solution by using either approach:

$$C^{BS}(S_t, \sigma^2, r, K, T - t) = S_t \Phi(d_1) - e^{-r(T-t)} K \Phi(d_2), \quad (14.7)$$

where $\Phi(\cdot)$ is the cdf of $N(0, 1)$ distribution, and

$$d_1 = \frac{1}{\sigma \sqrt{T-t}} \left[\log \frac{S_t}{K} + (r + \sigma^2/2)(T-t) \right], \quad (14.8)$$

$$d_2 = d_1 - \sigma \sqrt{T-t}. \quad (14.9)$$

Note:

(i) The equivalence between (14.4) and (14.5) was established via the Feynman-Kac formula (see the book by Karatzas and Shreve).

(ii) We will justify both (14.4) and (14.5) in the next lecture. Roughly speaking, (14.4) follows from a hedging argument, and the risk-neutral valuation principle for pricing derivatives behind (14.5) follows from the continuous-time Fundamental Theorems of Asset Pricing.

(iii) In the same spirit as in discrete-time finance, the discounted price $S_t^* = S_t/B_t$ forms a martingale under measure Q . Note that

$$S_t^* = S_0 \exp[(\mu - r - \sigma^2/2)t + \sigma W_t]. \quad (14.10)$$

Applying Itô's formula, we have

$$dS_t^*/S_t^* = (\mu - r) dt + \sigma dW_t. \quad (14.11)$$

Either (14.10) or (14.11) would imply that for S^* to be a martingale, it is necessary to have $\mu = r$.

14.3 Estimation of volatilities

To put the pricing formula (14.7) in action, numerical values of t, T, S_t, K, r, σ need to be specified. Among them, t, T, S_t, K are known or observable; while r and σ require more attention. r is assumed to be known now but will be considered as stochastic later in interest rate models.

Focusing on estimation of volatility σ , there are two substantially different methods:

(i) *Historical volatility*

This is just to estimate a population variance by a sample variance. Suppose we observe $S_{t_0}, S_{t_1}, \dots, S_{t_n}$ at $t_0 < t_1 < \dots < t_n$. Let

$$R_i = \log S_{t_i} - \log S_{t_{i-1}} \quad (14.12)$$

be the return incurred in $(t_{i-1}, t_i]$. Assume $t_i - t_{i-1} = \Delta, \forall i = 1, \dots, n$. Then R_1, \dots, R_n are iid normal random variables with mean $(\mu - \sigma^2/2)\Delta$ and variance $\sigma^2\Delta$. Naturally, a historical volatility estimate for σ is given by

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^n (R_i - \bar{R})^2}{(n-1)\Delta}} \quad \text{with} \quad \bar{R} = \frac{1}{n} \sum_{i=1}^n R_i.$$

Note that a common sense expression of return R_i , given by $(S_{t_i} - S_{t_{i-1}})/S_{t_{i-1}}$, is a first-order approximation of the definition in (14.12) with small Δ .

(ii) *Implied volatility*

An argument against the use of historical volatility is based on (a) volatility is not constant, but changes over time; (b) historical volatility only gives an estimate for the volatility over a past time period, but what is more relevant concerns the *market expectation* of the volatility in the near future. More specifically, if we want to determine the time t price of an option with maturity time T , then the market expectation of the volatility in the life time $(t, T]$ of the option should be incorporated consistently in the volatility estimation. One way to realize the market expectation is the following: Note that $C^{BS}(\cdot)$ in (14.7) is an invertible map of σ with other variables held fixed. Get a market price c_t from a “benchmark” call option written on the same stock $\{S_t\}$. Then find σ by inverting $C^{BS}(\cdot)$ from c_t . Such a value of σ is called the implied volatility.

Mathematically, implied volatility does not follow a traditional statistical inference guideline as historical volatility does. But the idea is far reaching. We will revisit it later when studying calibration of stochastic volatility (SV) models using both returns and option data.