Lecture 15 Girsanov Theorem and Risk Neutral Valuation

Assume the basic setting in a BS market: a riskless asset B following (14.1), a risky asset S following (14.2) with the Brownian filtration $\{\mathcal{F}_t\}$, defined in a probability space $(\Omega, \mathcal{F}, P)$.

15.1 Value processes, self-financing strategies, arbitrage

An adapted process $h = \{(h_{0t},\ h_{1t}) :\ 0 \le t \le T\}$ is called a dynamic portfolio in which $h_{0t}B_t$ represents the balance of bank account B at time t and h_{1t} is the number of shares of stock S at time t. In general, we should define *predictability* in continuous-time and require h to be a predictable process. However, it is not necessary here due to the continuous sample paths of Brownian motion W, i.e. if h is adapted to the Brownian filtration, then it is predictable.

$V = \{V_t\}$ is called the value process of a portfolio h where

$$V_t = h_{0t}\ B_t + h_{1t}\ S_t, \quad t \in [0, T]. \tag{15.1}$$

It is intuitive to call h a self-financing strategy if

$$dV_t = h_{0t}\ dB_t + h_{1t}\ dS_t \tag{15.2}$$

at every t. But a formal definition is required.

Definition 15.1 *Assume $\int_0^T E|h_{0t}|\ dt + \int_0^T Eh_{1t}^2\ dt < \infty$. h is called a self-financing strategy if*

$$V_t = V_0 + \int_0^t h_{0u}\ dB_u + \int_0^t h_{1u}\ dS_u \tag{15.3}$$

for $0 < t \le T$ with probability one.

Note:

(i) It follows from (14.1) and (14.2) that

$$\int_0^t h_{0u}\ dB_u = \int_0^t h_{0u}\ re^{ru}\ du$$

and

$$\int_0^t h_{1u}\ dS_u = \int_0^t h_{1u}\ S_u\ \mu\ du + \int_0^t h_{1u}\ S_u\ \sigma dW_u$$

for $0 < t \le T$.

(ii) Define the discounted value $V_t^* = V_t/B_t$. Then the self-financing condition can be given by

$$V_t^* = V_0 + \int_0^t h_{1u}\, dS_u^* \tag{15.4}$$

for $0 < t \le T$ with probability one. In fact,

$$\begin{aligned} dV_t^* &= -r\, V_t^*\, dt + e^{-rt} dV_t \\ &= -re^{-rt}(h_{0t}\, e^{rt} + h_{1t}\, S_t)\, dt + e^{-rt} h_{0t}\, de^{rt} + e^{-rt} h_{1t}\, dS_t \\ &= h_{1t}\, (-re^{-rt} S_t\, dt + e^{-rt} dS_t) \\ &= h_{1t}\, dS_t^*. \end{aligned}$$

Definition 15.2 *An arbitrage opportunity is said to exist if there is a self-financing strategy h such that its value process satisfies $V_0 = 0$, but $P(V_t \ge 0) = 1$ and $P(V_t > 0) > 0$ for some $t \in (0, T]$.*

We will define risk neutral probability measures [equivalent martingale measures (EMM)] and show that the existence of a risk neutral measure is a sufficient condition for no arbitrage.

15.2 Girsanov Theorem for a single Brownian motion

Recall that a probability measure Q on $(\Omega, \mathcal{F})$ is said to be absolutely continuous with respect to P, denoted by $Q \ll P$, if any $A \in \mathcal{F}$ with $P(A) = 0$ would imply $Q(A) = 0$. It is well-known in probability theory that $Q \ll P$ if and only if there exists a non-negative random variable Z such that $Q(A) = \int_A Z\, dP$ for all $A \in \mathcal{F}$. Z is called the Radon-Nikodým derivative (or density) of Q with respect to P, written as $Z = \frac{dQ}{dP}$. Moreover, P and Q are said to equivalent, denoted by $P \sim Q$, if both $Q \ll P$ and $P \ll Q$ hold. Note that $P \sim Q$ if and only if $P(Z > 0) = 1$.

Theorem 15.1 *Let $\{\theta_t\}$ be an adapted process satisfying $E[\exp(\int_0^T \theta_t^2\, dt/2)] < \infty$ (Novikov condition). For $0 \le t \le T$, define*

$$Z_t = \exp\left(-\int_0^t \theta_u\, dW_u - \int_0^t \theta_u^2\, du/2\right), \tag{15.5}$$

$$\widetilde{W}_t = W_t + \int_0^t \theta_u\, du, \tag{15.6}$$

and a measure Q with $\frac{dQ}{dP} = Z_T$. Then $\{Z_t\}$ is a martingale under P and $\{\widetilde{W}_t\}$ is a standard Brownian motion under Q.

The following martingale representation theorem is a basis for marketability of contingent claims in the BS market.

Theorem 15.2 *For every square-integrable martingale* $X = \{X_t\}_{t \in R_+}$ $(EX_t^2 < \infty$ *for every* $t)$ *with respect to the Brownian filtration, there is an adapted process* H *such that* $\int_0^T EH_t^2 \, dt < \infty$*, and*

$$X_t = X_0 + \int_0^t H_u \, dW_u, \quad t \in [0, T]. \tag{15.7}$$

See Karatzas and Shreve for the proofs of Theorem 15.1 and Theorem 15.2.

Corollary 15.1 *For a* $\mathcal{F}_T$*-measurable random variable* Y *with* $EY^2 < \infty$*, there is an adapted process* H *such that* $\int_0^T EH_t^2 \, dt < \infty$*, and*

$$Y = EY + \int_0^t H_u \, dW_u, \quad t \in [0, T]. \tag{15.8}$$

15.3 Risk neutral valuation in the BS market

A contingent claim Y, defined as a $\mathcal{F}_T$-measurable non-negative random variable with $EY^2 < \infty$, is said to be marketable if there is a self-financing dynamic portfolio h such that the value process V satisfies $V_T = Y$. The following result is a risk neutral valuation principle in the BS market.

Theorem 15.3 *In the BS market, every contingent claim* Y *is replicable by a portfolio* h *with the time-*t *value*

$$V_t = E_Q \left[e^{-r(T-t)} Y \mid \mathcal{F}_t \right], \tag{15.9}$$

where Q *is the risk-neutral measure defined by letting* $\theta_t = \frac{\mu - r}{\sigma}$ $\forall t \in [0, T]$ *in Theorem 15.1.*

Proof: Suppose Y is replicable by h with $V_T = Y$. It follows from (14.10) or (14.11) that setting $\theta = (\mu - r)/\sigma$ will make S^* a Q-martingale. By (15.4), V^* will be a Q-martingale, which with $V_T = Y$ will imply (15.9). To show Y is marketable, note that $X_t = E_Q(e^{-rT} Y | \mathcal{F}_t)$ forms a Q-martingale, Theorem 15.2 implies that

$$X_t = X_0 + \int_0^t H_u \, dW_u \tag{15.10}$$

for some adapted process $\{H_t\}$. Letting

$$h_{1t} = H_t / (\sigma S_t^*) \quad \text{and} \quad h_{0t} = X_t - h_{1t} S_t^* \tag{15.11}$$

will define a self-financing strategy with the value process

$$V_t = e^{rt} X_t = E_Q \left[e^{-r(T-t)} Y | \mathcal{F}_t \right], \tag{15.12}$$

and obviously $V_T = Y$.

The discussion in this lecture indicates that the BS market is a continuous-time analogue of the binomial tree model. We will study the Fundamental Theorems of Asset Pricing later.