

## Lecture 18 Exotic Options

The standard call and put options are usually called plain vanilla options. Many others, referred to as exotic options, are created these days and often traded over the counter. We will introduce several popular exotic options in this lecture. For simplicity, we assume the BS market although that is not necessary when defining those options in general.

### • Packages

A package is not really a new option, but a portfolio consisting of basic assets, call and put options. Such a linear combination can be easily valued by pricing separately its components. For example, a **collar option** is defined by the payoff

$$\xi_T = \min\{\max\{S_T, K_1\}, K_2\} = K_1 + (S_T - K_1)^+ - (S_T - K_2)^+$$

at expiration  $T$ , with constants  $K_2 > K_1 > 0$ . It is a portfolio of cash and two call options. Another example is a **break forward** which modifies a forward contract by setting the time- $T$  payoff

$$\xi_T = \max\{S_T, S_0 e^{rT}\} - K = (S_T - S_0 e^{rT})^+ + S_0 e^{rT} - K$$

with strike price  $K > S_0 e^{rT}$ . To make it comparable to a forward contract entered at time 0, the value of  $K$  is determined to make the break forward contract worthless at time 0, i.e.

$$K = e^{rT} [S_0 + C^{BS}(S_0, \sigma^2, r, S_0 e^{rT}, T).]$$

**Note:** What are the purposes, pros and cons of the collar options and break forwards?

### Digital options

A digital option (or called a binary option) is a contract with a discontinuous payoff function of the terminal underlying asset price. For instance, a “cash-or-nothing” call option has the terminal payoff  $DI_{\{S_T > K\}}$ , and a “cash-or-nothing” put option has the terminal payoff  $DI_{\{S_T < K\}}$  where  $D$  is a prescribed cash amount. An “asset-or-nothing” call (put resp.) option has the terminal payoff  $S_T I_{\{S_T > K\}}$  ( $S_T I_{\{S_T < K\}}$  resp.) A gap option has the terminal payoff  $(S_T - D) I_{\{S_T > K\}}$  (call) or  $(S_T - D) I_{\{S_T < K\}}$  (put). The risk neutral valuation principle applies to these digital options straightforwardly.

### • Forward-start options

With two dates  $T_0 < T$ , a forward-start option is a contract in which the holder receives, at time  $T_0$  (without additional cost), an option with expiration  $T$  and exercise price  $K = S_{T_0}$ . The holder must pay an upfront price at time 0. For a forward-start call option, the value at  $T_0$  should be

$$V_{T_0} = C^{BS}(S_{T_0}, \sigma^2, r, S_{T_0}, T - T_0) = S_{T_0} C^{BS}(1, \sigma^2, r, 1, T - T_0).$$

Hence for  $t \in [0, T_0]$ , the time- $t$  price is

$$V_t = S_t C^{BS}(1, \sigma^2, r, 1, T - T_0) = C^{BS}(S_t, \sigma^2, r, S_t, T - T_0).$$

### Compound options

A compound option is a standard option written on another standard option as the underlying asset. There are four types: call on call, put on call, call on put and put on put. Consider a call on call, i.e. a call option with strike price  $K_0$  and expiration  $T_0$  defined on a call with strike price  $K$  and expiration  $T = T_0 + \tau$  where  $\tau > 0$ . The time- $T_0$  payoff of is  $[C^{BS}(S_{T_0}, \sigma^2, r, K, \tau) - K_0]^+$  based on which the time- $t$  price of the compound option for  $t \in [0, T_0]$  can be expressed as  $E_Q\{e^{-r(T_0-t)}[C^{BS}(S_{T_0}, \sigma^2, r, K, \tau) - K_0]^+ | \mathcal{F}_t\}$ .

### • Barrier options

A barrier option is a contingent claim with the payoff

$$\xi_T = I_{\{\max_{u \leq T} S_u \geq a\}}$$

at expiration  $T$  where  $a > 0$  is a constant. Again, to ease the notation let us consider the time-0 price of the barrier option given  $S_0 = x$ :

$$V_0(x) = E_Q[e^{-rT} I_{\{\max_{t \leq T} S_t \geq a\}}] = e^{-rT} P\left(\max_{0 \leq t \leq T} W_t \geq a_t\right), \quad (18.1)$$

where  $a_t = \sigma^{-1}[\log a - \log x - (r - \sigma^2/2)t]$ . A useful strategy is to use the Girsanov transformation to “tilt” the straight line  $a_t$  to a horizontal line so we can apply the reflection principle in Proposition 11.1. Let  $\theta = -\sigma^{-1}(r - \sigma^2/2)$ . Then under  $P_\theta$ ,  $W$  is a Brownian motion with drift  $\theta$ . Define

$$\widetilde{W}_t = W_t - \theta t.$$

Then under  $P_\theta$ , the process  $\widetilde{W}$  will be (up to time  $T$ ) a standard Brownian motion with zero drift. Therefore,

$$P\left(\max_{0 \leq t \leq T} W_t \geq a_t\right) = E_\theta\{\exp[-\theta W_T + \theta^2 T/2] I_{\{\max_{0 \leq t \leq T} W_t \geq a_t\}}\} \quad (18.2)$$

$$= E_\theta\{\exp[-\theta \widetilde{W}_T - \theta^2 T/2] I_{\{\max_{0 \leq t \leq T} \widetilde{W}_t \geq b\}}\} \quad (18.3)$$

with  $b = \sigma^{-1}(\log a - \log x)$ , which can be calculated explicitly. (Exercise: Finish the calculation.)

- Other exotic options (**knock-in, knock-out, lookback, Asian, etc.**) will be discussed later if time permits.