

Lecture 2 Discrete Model Framework and Binomial Trees

2.1 Basic formulation

We start with a simple discrete model framework with several basic elements.

2.1.1 Asset price dynamics

- A finite sample space $\Omega = \{\omega_1, \dots, \omega_K\}$.
- A probability measure P on Ω with $P(\omega) > 0 \forall \omega \in \Omega$.
- A *filtration* $\mathbb{F} = \{\mathcal{F}_t, t = 0, 1, \dots, T\}$ with $\mathcal{F}_{t-1} \subseteq \mathcal{F}_t, t = 1, \dots, T$, where \mathcal{F}_t contains the information about the financial market available to the investors at time t . Usually, $t = 0, 1, \dots, T$ represent $T + 1$ trading dates. Since $T < \infty$, this is called a finite horizon model or a multiperiod model.
- A riskless *bank account* process $B = \{B(t), t = 0, 1, \dots, T\}$, where $B(0) = 1$ and $B(t) > 0 \forall t$. $B(t)$ is thought of as the time t value of a money market account when \$1 is deposited at time 0. Hence $B(t)$ is nondecreasing in t . Moreover, the quantity $r(t) = [B(t) - B(t-1)]/B(t-1)$ is thought of as the *interest rate* pertaining to the time interval $(t-1, t]$.
- N risky *security* processes $S_n = \{S_n(t), t = 0, 1, \dots, T\}, n = 1, \dots, N$, where $S_n(t) \geq 0$ is thought of as the time t price of risky security n (e.g. stock or bond).

Note that B, S_1, \dots, S_N are considered to be stochastic processes, i.e. for each $t, B(t), S_1(t), \dots, S_N(t)$ are all functions of ω . To ease the notation, the dependence on ω is usually not shown unless necessary. Furthermore, B, S_1, \dots, S_N are assumed to be *adapted* to the filtration \mathbb{F} . A stochastic process $\{X(t)\}$ is said to be adapted to the filtration \mathbb{F} if for each t , the random variable $X(t)$ is measurable with respect to \mathcal{F}_t , i.e. the information about $X(t)$ is contained in \mathcal{F}_t .

2.1.2 Trading strategies

A *trading strategy* $h = (h_0, h_1, \dots, h_N)$ is a vector of processes $h_n = \{h_n(t), t = 1, \dots, T\}, n = 0, 1, \dots, N$. Note that $h_n(0)$ is not specified, because for $n = 1, \dots, N, h_n(t)$ is interpreted as the number of units (e.g. shares of stock) that the investor owns (i.e. carries forward) from time $t-1$ to time t , whereas $h_0(t) B(t-1)$ represents the amount of money invested in the bank account at time $t-1$. A negative value of $h_n(t)$ corresponds to borrowing money from the bank (when $n = 0$) or selling short security n (when $n = 1, \dots, N$). h is also called a *portfolio*.

A trading strategy is a rule that specifies the investor's position in each security n at each time t and in each state of the world ω . In general, this rule should allow the investor to choose a position

in the securities based on the available information thus far without “looking into the future”. This is done by introducing the concept of *predictability*.

A stochastic process $\{X(t)\}$ is said to be predictable with respect to the filtration \mathbb{F} if for each $t = 1, 2, \dots$ the random variable $X(t)$ is measurable with respect to \mathcal{F}_{t-1} . (Note: “predictable” implies “adapted”, why?) In what follows we assume that each component of a trading strategy h is a predictable process.

2.1.3 Value processes, gain processes and self-financing strategies

The *value process* $V = \{V(t), t = 0, 1, \dots, T\}$ consists of the initial value of the portfolio

$$V(0) = h_0(1)B(0) + \sum_{n=1}^N h_n(1)S_n(0) \quad (2.1)$$

and the time t ($t \geq 1$) value of the portfolio

$$V(t) = h_0(t)B(t) + \sum_{n=1}^N h_n(t)S_n(t) \quad (2.2)$$

before any transactions are made at the same time. (Note: V is adapted, why?)

Denote $\Delta S_n(t) = S_n(t) - S_n(t-1)$ for the increment of S_n between $t-1$ and t . Then $h_n(t) \Delta S_n(t)$ represents the one-period gain or loss due to the ownership of $h_n(t)$ units of security n between $t-1$ and t ; and $\sum_{u=1}^t h_n(u) \Delta S_n(u)$ represents the cumulative gain or loss up to time t due to the investment of security n . Hence

$$G(t) = \sum_{u=1}^t h_0(u) \Delta B(u) + \sum_{n=1}^N \sum_{u=1}^t h_n(u) \Delta S_n(u) \quad (2.3)$$

represents the cumulative gain or loss of the portfolio up to time t . $G = \{G(t), t = 1, \dots, T\}$ is called a *gain process* (also adapted, why?).

A trading strategy is said to be *self-financing* if for $t = 1, \dots, T-1$,

$$V(t) = h_0(t+1) B(t) + \sum_{n=1}^N h_n(t+1) S_n(t). \quad (2.4)$$

The motivation is that the LHS represents the time t value of the portfolio just *before* any transactions (i.e. any changes of ownership positions) take place at that time, while the RHS represents the time t value of the portfolio right *after* any transactions (i.e. before the portfolio is carried forward to $t+1$). In general, the two values can be different, which means at time t some money is added to or withdrawn from the portfolio. However, for many applications this cannot happen

at other than $t = 0$ and $t = T$, and so it leads to the above definition. For a self-financing strategy, any change in the portfolio's value is due to a gain or loss in the investments.

It is straightforward to check (do it yourself) the following: A strategy h is self-financing if and only if

$$V(t) = V(0) + G(t), \quad t = 1, \dots, T. \quad (2.5)$$

Note that $V(1) = V(0) + G(1)$ always holds (why?).

2.1.4 Discounted prices

For the studies of finance modeling, what really matters is the behavior of the security prices relative to each other, rather than their absolute behavior. Hence we are interested in normalized versions of the security prices with respect to the price of a standard security — usually using the bank account for convenience. In general, some other riskless securities could be chosen as the “yardstick”, called the *numéraire*.

Define the *discounted price processes* $S_n^* = \{S_n^*(t), t = 0, 1, \dots, T\}$, $n = 1, \dots, N$ by

$$S_n^*(t) = S_n(t)/B(t), \quad t = 0, 1, \dots, T; \quad (2.6)$$

the *discounted value process* $V^* = \{V^*(t), t = 0, 1, \dots, T\}$ by

$$V^*(0) = h_0(1) + \sum_{n=1}^N h_n(1)S_n^*(0) \quad (2.7)$$

and

$$V^*(t) = h_0(t) + \sum_{n=1}^N h_n(t)S_n^*(t); \quad (2.8)$$

and the *discounted gain process* $G^* = \{G^*(t), t = 1, \dots, T\}$ by

$$G^*(t) = \sum_{n=1}^N \sum_{u=1}^t h_n(u) \Delta S_n^*(u), \quad t = 1, \dots, T. \quad (2.9)$$

Note in particular, $B^*(t) = 1$ and $\Delta B^*(t) = 0, \forall t$. It is also easy to check:

$$V^*(t) = V(t)/B(t), \quad t = 0, 1, \dots, T; \quad (2.10)$$

and that a strategy h is self-financing if and only if

$$V^*(t) = V^*(0) + G^*(t), \quad t = 1, \dots, T. \quad (2.11)$$

2.2 Binomial trees: an example

2.2.1 Illustration of concepts introduced in Section 2.1

- Figure 2.1, called a *binomial tree*, illustrates how a stock $S = \{S(t), t = 0, 1, 2, 3\}$ changes. We suppress the subscript n since $N = 1$. Assume $S(0) = \$2$, in each period the stock price either goes *up* by the factor $u = 1.07$ with probability $p = 0.6$, or goes *down* by the factor $d = 0.92$ with probability $1 - p = 0.4$, i.e. the moves over time are iid Bernoulli random variables. Hence $S(t) = S(0)u^{n_t}d^{t-n_t}$, $t = 0, 1, 2, 3$, where n_t represents the number of up moves up to t .

The sample space $\Omega = \{\omega_1, \dots, \omega_8\}$, where each ω_k corresponds to a path, e.g. ω_6 can be identified as the path “down-up-down” (dud), etc. Each probability $P(\omega_k)$ can be calculated easily, e.g. $P(\omega_6) = p(1 - p)^2 = 0.096$.

- Suppose the bank account process B is deterministic with a constant interest rate $r(t) \equiv 0.06$. In general, many different filtrations \mathbb{F} can be defined in which each \mathcal{F}_t contains the history of the stock up to t and perhaps some other information. This will become more useful later in this course. For now, we simply adopt the following particular filtration generated by the stock process S : Each \mathcal{F}_t involves precisely the history of S up to t and no additional information. More specifically, each \mathcal{F}_t is equivalent to a partition \mathcal{P}_t of Ω consisting of subsets of Ω with “no omission and no overlap”. The partitions can be specified by paths:

$$\begin{aligned} \mathcal{P}_1 &= \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\} \\ &= \{\{uuu, uud, udu, udd\}, \{duu, dud, ddu, ddd\}\}; \end{aligned}$$

$$\begin{aligned} \mathcal{P}_2 &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8\}\} \\ &= \{\{uuu, uud\}, \{udu, udd\}, \{duu, dud\}, \{ddu, ddd\}\}; \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}_3 &= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\} \\ &= \{\{uuu\}, \{uud\}, \{udu\}, \{udd\}, \{duu\}, \{dud\}, \{ddu\}, \{ddd\}\}. \end{aligned}$$

As t increases, the partition \mathcal{P}_t becomes finer and \mathcal{F}_t reveals more information about the evolution of stock S .

- The value process, gain process and their discounted versions depend on a given trading strategy (portfolio process). For each t , $B(t) = (1 + 0.06)^t$ and the portfolio is $(h_0(t), h_1(t))$.

Following (2.1) and (2.2), we have the value process

$$V(0) = h_0(1) + 2.00 h_1(1),$$

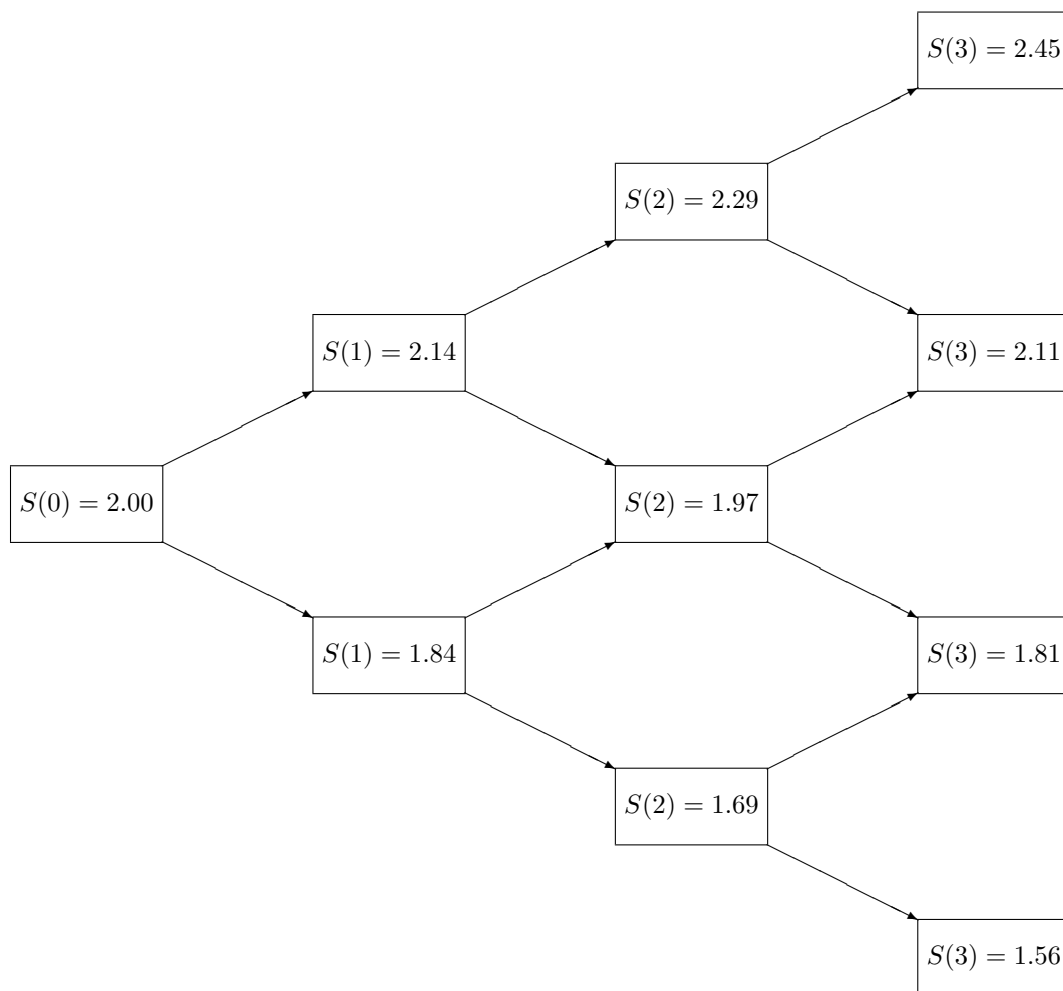


Figure 2.1: Stock price tree

$$V(1) = \begin{cases} (1 + 0.06) h_0(1) + 2.14 h_1(1), & \text{on } \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ (1 + 0.06) h_0(1) + 1.84 h_1(1), & \text{on } \{\omega_5, \omega_6, \omega_7, \omega_8\} \end{cases}$$

$$V(2) = \begin{cases} (1 + 0.06)^2 h_0(2) + 2.29 h_1(2), & \text{on } \{\omega_1, \omega_2\} \\ (1 + 0.06)^2 h_0(2) + 1.97 h_1(2), & \text{on } \{\omega_3, \omega_4\} \text{ or } \{\omega_5, \omega_6\} \\ (1 + 0.06)^2 h_0(2) + 1.69 h_1(2), & \text{on } \{\omega_7, \omega_8\} \end{cases}$$

and

$$V(3) = \begin{cases} (1 + 0.06)^3 h_0(3) + 2.45 h_1(3), & \text{on } \{\omega_1\} \\ (1 + 0.06)^3 h_0(3) + 2.11 h_1(3), & \text{on } \{\omega_2\} \text{ or } \{\omega_3\} \text{ or } \{\omega_5\} \\ (1 + 0.06)^3 h_0(3) + 1.81 h_1(3), & \text{on } \{\omega_4\} \text{ or } \{\omega_6\} \text{ or } \{\omega_7\} \\ (1 + 0.06)^3 h_0(3) + 1.56 h_1(3), & \text{on } \{\omega_8\}. \end{cases}$$

The gain process in (2.3) can be written (in this example) as

$$G(t) = G(t-1) + h_0(t) \Delta B(t) + h_1(t) \Delta S(t).$$

Hence we have

$$G(1) = \begin{cases} 0.06 h_0(1) + 0.14 h_1(1), & \text{on } \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ 0.06 h_0(1) - 0.16 h_1(1), & \text{on } \{\omega_5, \omega_6, \omega_7, \omega_8\} \end{cases}$$

$$G(2) = \begin{cases} 0.06 h_0(1) + 0.14 h_1(1) + 0.06 h_0(2) + 0.15 h_1(2) & \text{on } \{\omega_1, \omega_2\} \\ 0.06 h_0(1) + 0.14 h_1(1) + 0.06 h_0(2) - 0.17 h_1(2) & \text{on } \{\omega_3, \omega_4\} \\ 0.06 h_0(1) - 0.16 h_1(1) + 0.06 h_0(2) + 0.13 h_1(2) & \text{on } \{\omega_5, \omega_6\} \\ 0.06 h_0(1) - 0.16 h_1(1) + 0.06 h_0(2) - 0.15 h_1(2) & \text{on } \{\omega_7, \omega_8\} \end{cases}$$

and

$$G(3) = \begin{cases} 0.06 h_0(1) + 0.14 h_1(1) + 0.06 h_0(2) + 0.15 h_1(2) \\ \quad + 0.07 h_0(3) + 0.16 h_1(3), & \text{on } \{\omega_1\} \\ 0.06 h_0(1) + 0.14 h_1(1) + 0.06 h_0(2) + 0.15 h_1(2) \\ \quad + 0.07 h_0(3) - 0.18 h_1(3), & \text{on } \{\omega_2\} \\ 0.06 h_0(1) + 0.14 h_1(1) + 0.06 h_0(2) - 0.17 h_1(2) \\ \quad + 0.07 h_0(3) + 0.14 h_1(3), & \text{on } \{\omega_3\} \\ 0.06 h_0(1) + 0.14 h_1(1) + 0.06 h_0(2) - 0.17 h_1(2) \\ \quad + 0.07 h_0(3) - 0.16 h_1(3), & \text{on } \{\omega_4\} \\ 0.06 h_0(1) - 0.16 h_1(1) + 0.06 h_0(2) + 0.13 h_1(2) \\ \quad + 0.07 h_0(3) + 0.14 h_1(3), & \text{on } \{\omega_5\} \\ 0.06 h_0(1) - 0.16 h_1(1) + 0.06 h_0(2) + 0.13 h_1(2) \\ \quad + 0.07 h_0(3) - 0.16 h_1(3), & \text{on } \{\omega_6\} \\ 0.06 h_0(1) - 0.16 h_1(1) + 0.06 h_0(2) - 0.15 h_1(2) \\ \quad + 0.07 h_0(3) + 0.12 h_1(3), & \text{on } \{\omega_7\} \\ 0.06 h_0(1) - 0.16 h_1(1) + 0.06 h_0(2) - 0.15 h_1(2) \\ \quad + 0.07 h_0(3) - 0.13 h_1(3), & \text{on } \{\omega_8\}. \end{cases}$$

We now look at the condition (2.4) for self-financing portfolios. For $t = 1$,

$$\begin{aligned} 1.06 h_0(1) + 2.14 h_1(1) &= 1.06 h_0(2) + 2.14 h_1(2), & \text{on } \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ 1.06 h_0(1) + 1.84 h_1(1) &= 1.06 h_0(2) + 1.84 h_1(2), & \text{on } \{\omega_5, \omega_6, \omega_7, \omega_8\}. \end{aligned}$$

For $t = 2$,

$$\begin{aligned} 1.12 h_0(2) + 2.29 h_1(2) &= 1.12 h_0(3) + 2.29 h_1(3), & \text{on } \{\omega_1, \omega_2\} \\ 1.12 h_0(2) + 1.97 h_1(2) &= 1.12 h_0(3) + 1.97 h_1(3), & \text{on } \{\omega_3, \omega_4\} \text{ or } \{\omega_5, \omega_6\} \\ 1.12 h_0(2) + 1.69 h_1(2) &= 1.12 h_0(3) + 1.69 h_1(3), & \text{on } \{\omega_7, \omega_8\}. \end{aligned}$$

In general, there are many trading strategies that satisfy the specified self-financing conditions.

2.2.2 What is a fair price?

Suppose at $t = 0$ you want to evaluate a contract, called a *call option*, that involves the future stock price: at $T = 3$, you have the option of either buying the stock for \$2.05 or not buying it. The call option assures you a “no loss” outcome at $T = 3$, i.e. your payoff would be $(S(3) - 2.05)^+$. Thus the option should bear a fair price (or called the value of the option) at $t = 0$. What should the fair price be?

To answer the question, we use the backward induction: first consider a one-step evolution at the upper right corner of the binomial tree; then extend the result to the entire tree.

Starting from $S(2) = 2.29$, a portfolio $(h_0(3), h_1(3))$ has the value $1.06^2 h_0(3) + 2.29 h_1(3)$, then becomes either $1.06^3 h_0(3) + 2.45 h_1(3)$ or $1.06^3 h_0(3) + 2.11 h_1(3)$ at $T = 3$. If we set

$$\begin{cases} 1.06^3 h_0(3) + 2.45 h_1(3) = 2.45 - 2.05 \\ 1.06^3 h_0(3) + 2.11 h_1(3) = 2.11 - 2.05 \end{cases} \quad (2.12)$$

then the solution $h_0(3) = -1.723$, $h_1(3) = 1$ specifies an investment strategy at $S(2) = 2.29$ that leads to the same payoff as the option, no matter what the outcome of $S(3)$ may be. The value of this (one-step) portfolio, $-1.723 \cdot 1.06^2 + 1 \cdot 2.29 = 0.36$, can be taken as a fair price of the option at $t = 2$ with $S(2) = 2.29$. The following *arbitrage* argument explains why, provided we assume that any opportunity to make a riskless profit (called an arbitrage opportunity) is ruled out.

Denote the option price by \mathcal{P} . If $\mathcal{P} < 0.36$, then at $t = 2$ a clever investor can buy the option for \mathcal{P} and in the meantime follow the strategy $h_0(3) = 1.723$, $h_1(3) = -1$, i.e. opposite to what was derived in the last paragraph. This amounts to *short selling* one share of stock (\$2.29) (See Hull’s book for how to implement the short-selling.) and deposit $1.723 \cdot \$1.06^2$ in the bank. At $T = 3$, the amount collected from the option is exactly what is needed to settle the obligation associated with the portfolio. Hence the investor could lock into a riskless profit of $0.36 - \mathcal{P}$. On the other hand, if $\mathcal{P} > 0.36$, then the investor would sell short the option for \mathcal{P} and follow the

strategy $h_0(3) = -1.723$, $h_1(3) = 1$, i.e. to borrow $1.723 \cdot \$1.06^2$ from the bank and buy one share of stock ($\$2.29$). At $T = 3$, the value of the portfolio matches the exact obligation with the option in every possible state of nature. This has the investor lock into a riskless profit of $\mathcal{P} = 0.36$.

Therefore, the fair price of the option (or the value of the option) at $S(2) = 2.29$ is 0.36 ; By the same token, values of the option at two other $S(2)$ can be derived; Moving one step back, values of the option at two $S(1)$ can be specified; Finally, we end up with the value of the option at $S(0)$. The calculation yields Figure 2.2, a binomial tree for option values $\{V(t), t = 0, 1, 2, 3\}$ (Check it yourself).

In particular, the fair price of the call option at $t = 0$ is $\$0.28$.

A by-product is Figure 2.3, a portfolio binomial tree. The pair in each box represents the (h_0, h_1) that applies to the two branches connecting the box and its two “descendants”. For example, $h_0(2) = -0.27$ and $h_1(2) = 0.179$ indicate that at $t = 1$, the amount $0.27 \cdot \$1.06$ is borrowed from the bank and 0.179 shares of stocks is bought with the unit price $\$1.84$. This portfolio is held until the next transaction at $t = 2$.

2.2.3 Risk neutral probabilities

More information can be extracted from this example. Let

$$q = \frac{1 + r - d}{u - d} = \frac{1.06 - 0.92}{1.07 - 0.92} = \frac{14}{15}, \quad (2.13)$$

and denote by $V(t, k)$ the value of the call option at the location (t, k) . Note that the location of each box in this recombining tree is uniquely identified by a pair (t, k) with $n_t = k$. Then we have

$$V(t, k) = (1 + r)^{-1} [q V(t + 1, k + 1) + (1 - q) V(t + 1, k)]. \quad (2.14)$$

Hence the value in each box is expressed as a discounted weighted average of the values in the two descendent boxes. When the factors u , d and the rate r are constants over the tree as in this example, so is the weight q . Such a binomial tree is called a homogeneous tree. Later on we will demonstrate that the method extends to inhomogeneous trees also.

Hence we have two methods to price the call option: one by solving equations like (2.12) thus replicating the portfolio; the other by taking discounted weighted averages like (2.14). The impact is far-reaching. If we define

$$Q(\omega) = q^{U(\omega)}(1 - q)^{3-U(\omega)}, \quad (2.15)$$

where $U(\omega)$ represents the total number of up moves in the path ω , then Q is a probability measure on Ω , called a *risk neutral probability measure*, e.g. $Q(\omega_3) = q^2(1 - q) = 0.058$.

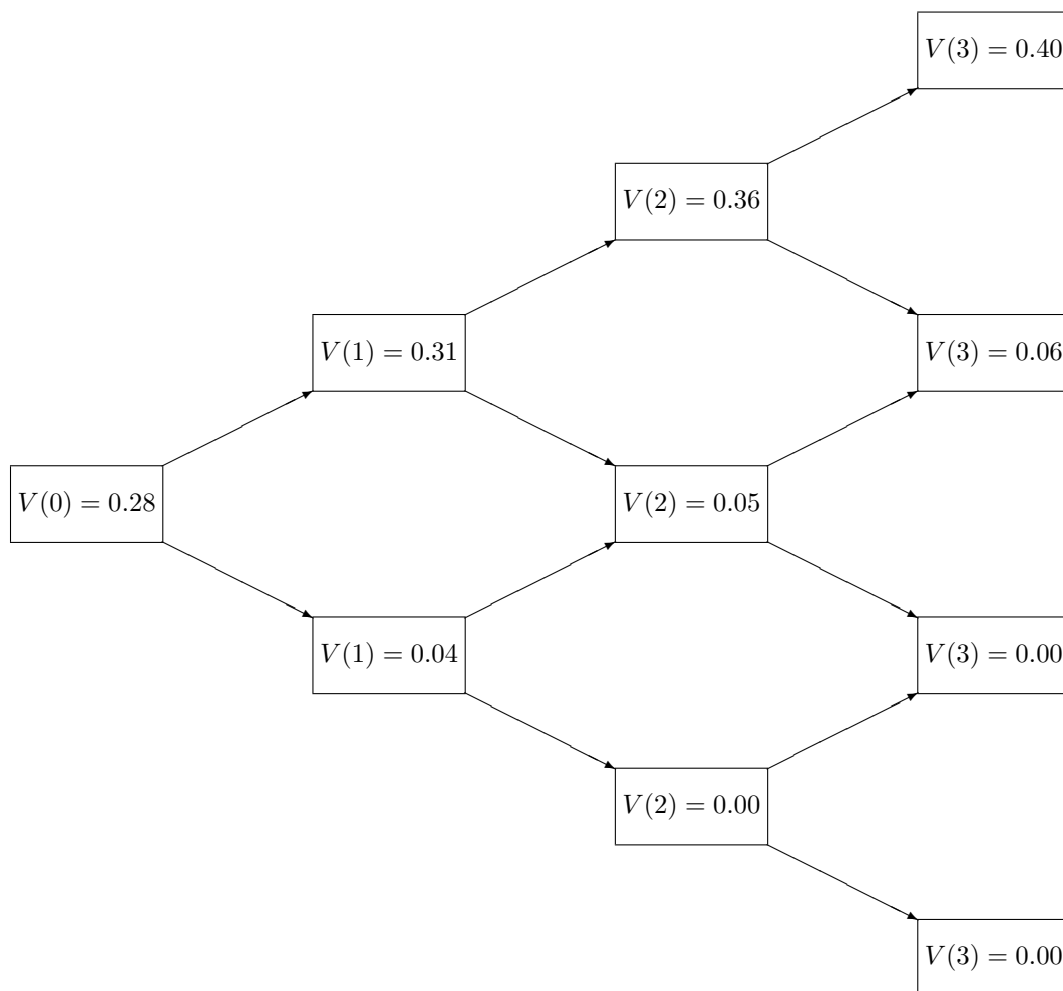


Figure 2.2: Option value tree

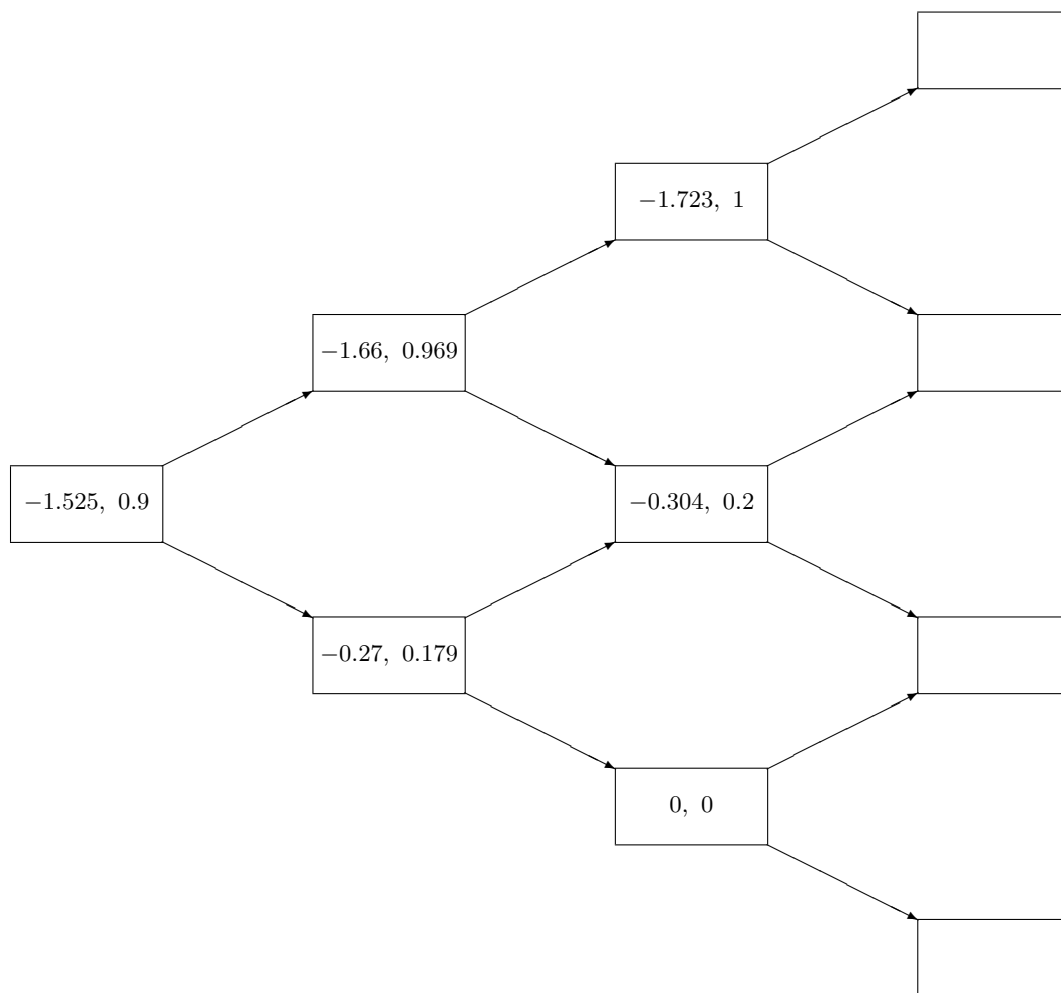


Figure 2.3: Portfolio tree

It is interesting to notice that the underlying Bernoulli probability p and the probability measure P were not relevant in the option pricing. It is the risk neutral probability factor q and measure Q that are useful. In general, $p \neq q$ and $P \neq Q$. The probability measure Q is not a part of the model assumptions, but constructed from the market data — stocks and interest rates. In that sense, Q is an empirical measure.

Exercises:

2.1 Calculate the risk neutral probabilities $Q(\omega_k)$, $k = 1, \dots, 8$.

2.2 Check whether the weighted averages produce the same results in option pricing as in Figure 2.2. Some discrepancies may be due to rounding errors.

2.3 If we change the interest rate from 6% to 8%, what would happen? Can you still carry out all the calculation? Why?