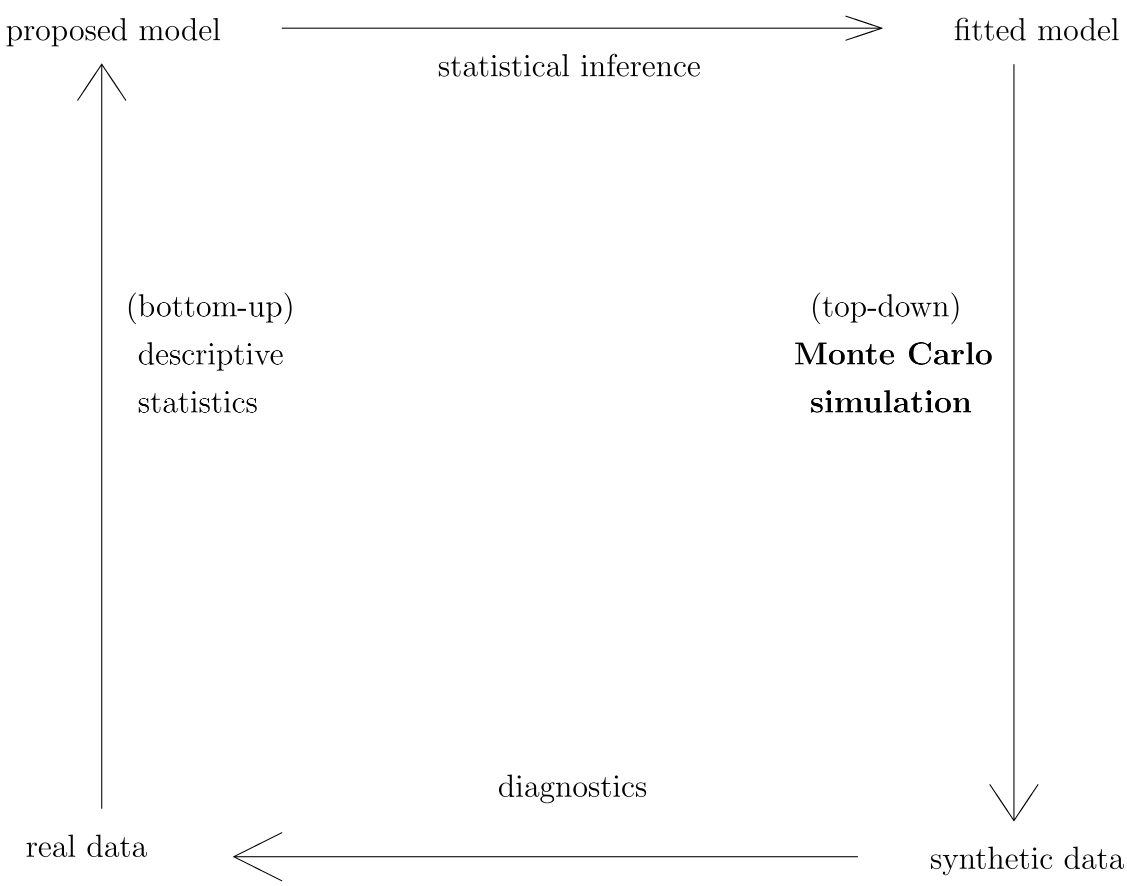
Lecture 21 Monte Carlo Methods in Finance

21.1 What Monte Carlo for?

Monte Carlo simulation, considered as mathematical experimentation, can help in statistical modeling, optimization for a function of many variables, numerical integration over a high-dimensional space, and a lot more ...

Example 21.1: Statistical modeling



See the diagram of "reconstruction cycle". Consider the "usual" procedure: Given a data set, summarize it, propose a model, fit the model (statistical inference), generate synthetic data from the fitted model (this is the simulation step!), compare the simulated data with the real one. May need to start over again, etc. Such cases are plenty: regression, time series, spatial stat, with many applications. Why need a model? For prediction, better understanding, generalization, etc.

Example 21.2: Optimization

Maybe the last resort when calculus, discrete search, etc. cannot do it. Optimization via simulation is usually very time-consuming. Consider the 2D Ising model in statistical physics. Minimizing the energy function $U(x)$ is equivalent to finding the ground state $\hat{x}$, i.e. the mode of the Gibbs distribution. This can be done by simulated annealing (SA). See Liu's book, sec. 1.3 for the Ising model, sec. 10.2 for SA. Geman and Geman (1984) applies SA to image analysis.

Example 21.3: Numerical integration

Consider the integral

$$I = \int_D g(x)\pi(x)dx \tag{21.1}$$

where the function g and probability density π are defined over a domain $D \subset I\!R^d$. How to compute I numerically?

If using Monte Carlo simulation, we generate iid samples $X^{(1)}, \ldots, X^{(n)}$ from π and approximate I by

$$I_n = \frac{1}{n}\sum_{i=1}^{n} g(X^{(i)}). \tag{21.2}$$

It is well known that I_n is unbiased, i.e. its expectation $E_\pi I_n = I$, and I_n converges to I with π-probability one as $n \to \infty$ (LLN). The approximation error is of order $O(n^{-1/2})$. This follows from CLT, or can be viewed through the mean square error (MSE)

$$\sqrt{E_\pi(I_n - I)^2} = \sqrt{Var_\pi I_n} = \frac{\sigma}{\sqrt{n}},$$

where σ is the standard deviation of $g(X^{(1)})$. Note that the order $O(n^{-1/2})$ does not depend on the dimension d, although the constant σ does (variance reduction is a major issue in Monte Carlo simulation).

In contrast, the efficiency of deterministic numerical integration methods (lattice approximations) is critically affected by the dimension d. More precisely, assume the functions g and π are smooth enough and approximate I by a sum involving $g(x)\pi(x)$ evaluated at n nodal points in D. Let δ be the step size (spacing) in the lattice approximation. Then $\delta = O(n^{-1/d})$. Suppose a given method (e.g. Trapezoidal rule, Simpson's rule, etc. see *Numerical Recipes in C*, 2nd edition, sec. 4.1) yields an error rate $O(\delta^k) = O(n^{-k/d})$ for some positive integer k, which gets worse with greater d.

Therefore, Monte Carlo integration becomes more efficient than deterministic numerical methods if (and only if) the dimension d is high.

21.2 Pricing American options by least squares Monte Carlo (LSMC)

For a simple illustration purpose, we focus on an example in which a single stock evolves over 8 possible sample paths through $T = 3$ discrete time periods. Assume a bank account follows continuous compounding with constant rate $r = 6\%$, i.e. the growth factor in each time period is e^r. Consider an American put option. At time $t = 1, 2$, the holder can decide whether to exercise and gain an immediate pay-off $(K - S_t)^+$ or continue, and at $T = 3$, the choice would be between "exercise" or "not exercise". Here is an algorithm, called LSMC, proposed in Longstaff and Schwartz (2001) (I will email you the paper): Let Y be a generic symbol for the discounted pay-off of the option with "continuation", and X be the state variable (stock). Note that both X and Y are time-dependent. Denote the conditional expectation operator under a risk neutral measure Q by $E(\cdot)$. At each time $t < T$, we compare $E(Y|X)$ to the pay-off $(K - S_t)^+$ and choose between "continuation" and "stop and exercise". The idea is to approximate $E(Y|X)$ by a simple function of X, e.g. a quadratic polynomial $a_0 + a_1X + a_2X^2$, where the coefficients a_0, a_1, a_2 are determined at each step by least squares estimates in a backward induction procedure. In what follows, we assume the strike price $K = 1.10$. The numerical results in 9 concrete steps are contained in Tables 21.1 — 21.9. Here is a summary.

- Table 21.1 contains 8 stock price sample paths.
- Table 21.2 is for $(K - S_3)^+$ at $T = 3$, i.e. they represent the cash flows that would be received if the option were European rather than American.
- Table 21.3 concerns only 5 effective sample paths where the put is in the money at $t = 2$. The least squares estimates yield the regression $E(Y|X) = -1.070 + 2.983\ X - 1.813\ X^2$.
- In Table 21.4, the "Exercise" column follows $1.10 - X$ and the "Continuation" column follows the fitted regression.
- Table 21.5 follows Table 21.4.
- Table 21.6 is obtained by using a different method from the one used in Table 21.3 to avoid certain upward bias (see the paper for details). There are 5 effective sample paths indexed 1,4,6,7,8 with in the money option at $t = 1$. The actual realized cash flow over each of them (see Table 21.5) is used in calculating Y, instead of the conditional expectation of Y estimated at $t = 2$. Since the option can only be exercised once, future cash flows occur at either $t = 2$ or $T = 3$, but not both. The least squares regression at $t = 1$ is given by $E(Y|X) = 2.038 - 3.335\ X + 1.356\ X^2$.
- Table 21.7 follows Table 21.6.
- Table 21.8: obvious.

- Table 21.9: Only the "1" entries in Table 21.8 yield nonzero entries here.

Path	$t=0$	$t=1$	$t=2$	$t=3$
ω_1	1.00	1.09	1.08	1.34
ω_2	1.00	1.16	1.26	1.54
ω_3	1.00	1.22	1.07	1.03
ω_4	1.00	0.93	0.97	0.92
ω_5	1.00	1.11	1.56	1.52
ω_6	1.00	0.76	0.77	0.90
ω_7	1.00	0.92	0.84	1.01
ω_8	1.00	0.88	1.22	1.34

Table 21.1: Stock price sample paths

Path	$t=1$	$t=2$	$t=3$
ω_1	—	—	0
ω_2	—	—	0
ω_3	—	—	0.07
ω_4	—	—	0.18
ω_5	—	—	0
ω_6	—	—	0.20
ω_7	—	—	0.09
ω_8	—	—	0

Table 21.2: Cash flow $(K - S_3)^+$ at $t = 3$ with $K = 1.10$

Path	Y	X
ω_1	$0 \times e^{-r}$	1.08
ω_2	—	—
ω_3	$0.07 \times e^{-r}$	1.07
ω_4	$0.18 \times e^{-r}$	0.97
ω_5	—	—
ω_6	$0.20 \times e^{-r}$	0.77
ω_7	$0.09 \times e^{-r}$	0.84
ω_8	—	—

Table 21.3: Regression at $t = 2$ with discount factor $e^{-r} = 0.94176$

Path	Exercise	Continuation
ω_1	0.02	0.0369
ω_2	—	—
ω_3	0.03	0.0461
ω_4	0.13	0.1176
ω_5	—	—
ω_6	0.33	0.1520
ω_7	0.26	0.1565
ω_8	—	—

Table 21.4: Exercise at $t = 2$ or continue ?

Path	$t = 1$	$t = 2$	$t = 3$
ω_1	—	0	0
ω_2	—	0	0
ω_3	—	0	0.07
ω_4	—	0.13	0
ω_5	—	0	0
ω_6	—	0.33	0
ω_7	—	0.26	0
ω_8	—	0	0

Table 21.5: Cash flow at $t = 2$ and $t = 3$

Path	Y	X
ω_1	$0 \times e^{-r}$	1.09
ω_2	—	—
ω_3	—	—
ω_4	$0.13 \times e^{-r}$	0.93
ω_5	—	—
ω_6	$0.33 \times e^{-r}$	0.76
ω_7	$0.26 \times e^{-r}$	0.92
ω_8	$0 \times e^{-r}$	0.88

Table 21.6: Regression at $t = 1$ with discount factor $e^{-r} = 0.94176$

Path	Exercise	Continuation
ω_1	0.01	0.0139
ω_2	—	—
ω_3	—	—
ω_4	0.17	0.1092
ω_5	—	—
ω_6	0.34	0.2866
ω_7	0.18	0.1175
ω_8	0.22	0.1533

Table 21.7: Exercise at $t = 1$ or continue ?

Path	$t = 1$	$t = 2$	$t = 3$
ω_1	0	0	0
ω_2	0	0	0
ω_3	0	0	1
ω_4	1	0	0
ω_5	0	0	0
ω_6	1	0	0
ω_7	1	0	0
ω_8	1	0	0

Table 21.8: Optimal stopping rule with 0 =“not exercise” and 1 = “exercise”

Path	$t = 1$	$t = 2$	$t = 3$
ω_1	0	0	0
ω_2	0	0	0
ω_3	0	0	0.07
ω_4	0.17	0	0
ω_5	0	0	0
ω_6	0.34	0	0
ω_7	0.18	0	0
ω_8	0.22	0	0

Table 21.9: Optimal cash flow matrix

A number of related issues will be discussed in the next lecture.