Lecture 22 More on Monte Carlo Methods in Finance

Monte Carlo methods apply to several aspects in mathematical finance, which we will briefly describe in what follows.

1 Simulation of SDEs

Consider an Itô process X defined as a solution to the SDE

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t, \ t \in [0, T], \quad \text{with} \quad X_0 = x. \tag{22.1}$$

For simplicity, assume $\mu_t = a(X_t)$ and $\sigma_t = b(X_t)$ for some deterministic functions $a(\cdot)$ and $b(\cdot)$. The Euler discretization scheme (see [2] or [3]) provides a first-order approximation to (22.1) as follows: For a small time increment $\Delta > 0$, suppose $T/\Delta = n$ (a positive integer). Let $Y_i = X_{i\Delta}$, $i = 0, 1, ..., n$. Then (22.1) is approximated by the (discrete) time series

$$Y_i = Y_{i-1} + a(Y_{i-1}) \, \Delta + b(Y_{i-1}) \, \sqrt{\Delta} \, \epsilon_i, \quad i = 1, ..., n, \quad \text{with} \quad Y_0 = x, \tag{22.2}$$

where $\epsilon_1, \epsilon_2, ...$ are iid $N(0, 1)$ random variables. An obvious way to simulate the sequence $\{Y_i\}$ would start with generating the sequence $\{\epsilon_i\}$ then follow the iterations in (22.2).

What is the discretization error when using $\{Y_i\}$ as a proxy of $\{X_t\}$? It can be shown that under certain conditions on $a(\cdot)$ and $b(\cdot)$, we have an upper bound for the mean square error of sample paths

$$E\left(\sup_{t\in[0,T]} |Y_{[t/\Delta]} - X_t|^2\right) \leq C \, \Delta \tag{22.3}$$

for some constant $C > 0$ that depends $a(\cdot)$, $b(\cdot)$ and T. Note that (22.1) and (22.2) can be P-dynamics or Q-dynamics, and the expectation in (22.3) would be under P or Q accordingly.

2 Computation of option prices

For most (European) option pricing problems, the computation amounts to numerical PDEs or numerical integration via Monte Carlo simulation. By time shifting, let us consider the time-0 price of an European option $g(X_T)$ with maturity T, expressed as

$$P_0(x) = E_Q \left[e^{-rT} g(X_T) \; | X_0 = x \right], \tag{22.4}$$

where X_t, $t \in [0, T]$ is the time-t value of an underlying risky asset whose Q-dynamics follows (22.1). Here is an algorithm (we call it "brute-force" Monte Carlo) that yields $P_0(x)$:

Step 1: Generate M independent sample paths of $Y = \{Y_i\}$, denoted by $Y^{(m)}$, $m = 1, ..., M$, i.e. we need to generate M independent processes $\epsilon^{(m)}$, $m = 1, ..., M$, with each $\epsilon^{(m)}$ being a sequence of n iid $N(0, 1)$ random variables, then iterate following the Q-dynamics governed by (22.2).

Step 2: On each path $Y^{(m)}$, calculate $g(Y_n^{(m)})$.

Step 3: Calculate the average

$$P_0(x; \Delta, M) = \frac{1}{M} \sum_{m=1}^{M} e^{-rT} g(Y_n^{(m)}) \tag{22.5}$$

as an approximation for $P_0(x)$.

Note:

(i) Since only the terminal variable $Y_n^{(m)}$ is needed in Step 2 for this problem, we need not generate the entire path $Y^{(m)}$ if we are able to produce $Y_n^{(m)}$ directly by some other method, e.g. the transition density of Y_n given $Y_0 = x$ might be available in certain cases. However, many other interesting applications require the realizations of Y_i, $i = 1, ..., n$.

(ii) Given a limited computational time and a prescribed upper bound for the error, say

$$E[|P_0(x; \Delta, M) - P_0(x)|^2], \tag{22.6}$$

there is a trade-off between a small Δ and a large M. [1] suggested a guideline of $M = O(\Delta^{-2})$ for the Euler scheme and showed why it is reasonable. Following this guideline, we should simulate more than $10,000$ independent paths for a financial time series of 100 daily prices.

More complicated derivatives, such as path-dependent options and American options, can be computed by similar Monte Carlo schemes, but would require greater computational intensity.

3 MCMC calibration of SV models

The above brute-force Monte Carlo works for computing a small number of derivative prices, but would become intractable if such numerical integration need to be performed a large number of times. See Section 20.2.3 in Lecture 20 for details. We will give a power point presentation to highlight the challenges and suggest certain alternative probability approximation schemes.

References

[1] Duffie, D. and Glynn, P. (1995). Efficient Monte Carlo simulation of security prices. *Annals of Applied Probability* **5**, 897-905.

[2] Glasserman, P. (2000). *Monte Carlo Methods in Financial Engineering.* Springer-Verlag.

[3] Kloeden, P. E., and Platen, E. (1992). *Numerical Solution of Stochastic Differential Equations.* Springer-Verlag.