

Lecture 3 Fundamental Theorems of Asset Pricing

3.1 Arbitrage and risk neutral probability measures

Several important concepts were illustrated in the example in Lecture 2:

- arbitrage;
- risk neutral probability measures;
- contingent claims such as call options;
- two different ways to price a contingent claim.

Now begin our general studies on these topics. Lecture 3 contains two fundamental theorems of asset pricing. Theorem 3.1 concerns the equivalence between no arbitrage and existence of risk neutral probability measures; and Theorem 3.2 concerns the equivalence between market completeness and uniqueness of the risk neutral measure. We will demonstrate the valuation of a contingent claim by replicating portfolios or taking conditional expectations with respect to a risk neutral probability measure (or called an *equivalent martingale measure*).

An *arbitrage opportunity* is said to exist if there is a self-financing strategy h whose value function satisfies

- (a) $V(0) = 0$;
- (b) $V(T) \geq 0$;
- (c) $P(V(T) > 0) > 0$.

Although a smart investor may seek and grab such a *riskless* way of making a profit, it would only be a transient opportunity. Once more investors and traders jump in to share the “free lunch”, prices of the securities would change immediately. Hence the old equilibrium would break down and be replaced by a new equilibrium, i.e. arbitrage opportunities would vanish. That is why we assume no arbitrage. It is also an implication of the efficient market hypothesis.

Example 3.1 In the example in Lecture 2, suppose the constant interest rate is 8%. Then an arbitrage opportunity can be found easily. Just do nothing at $t = 0$ and $t = 1$, and short sell a number of shares of the stock (if allowed) at $t = 2$, deposit the proceeds in the bank account, close the short position at $T = 3$ (buy back the same number of shares of the stock and return them). This enables the investor to make a net profit at $T = 3$. (Convince yourself this strategy is self-financing and creates an arbitrage.)

Example 3.2 Let the interest rate equal 7%. The situation is similar to but slightly more interesting than Example 3.1. Try to find an arbitrage strategy.

In general, it is not always easy to check directly whether an arbitrage opportunity exists. A useful criterion is given via equivalent martingale measures.

Assume the framework in Section 2.1. A stochastic process $X = \{X(t), t = 0, 1, \dots, T\}$ is called a martingale under a probability measure Q and with respect to a filtration \mathbb{F} , if the conditional expectation

$$E_Q (X(t) | \mathcal{F}_{t-1}) = X(t-1) \quad \forall t = 1, \dots, T.$$

Sometimes we call X a Q -martingale.

Theorem 3.1 (*First Fundamental Theorem of Asset Pricing*) *No arbitrage \iff there is a probability measure Q with $Q(\omega) > 0 \forall \omega \in \Omega$, such that every discounted price process $S_n^* = \{S_n^*(t), t = 0, 1, \dots, T\}$ is a Q -martingale, $n = 1, \dots, N$. Such a measure Q is called an equivalent martingale measure (EMM).*

We follow the approach due to Harrison and Pliska given in their seminal paper (1981, *Stoch. Proc. and Their Appl.* **11**, 215-260).

Proof of sufficiency “ \Leftarrow ”

This is an easy direction. It suffices to verify that $\{G^*(t)\}$ is Q -martingale [so is $\{V^*(t)\}$ by (2.11)]. Note that by (2.9), for every $t = 1, \dots, T$, the conditional expectation under Q is

$$E [\Delta G^*(t) | \mathcal{F}_{t-1}] = \sum_{n=1}^N E [h_n(t) \Delta S_n^*(t) | \mathcal{F}_{t-1}] = \sum_{n=1}^N h_n(t) E [\Delta S_n^*(t) | \mathcal{F}_{t-1}] = 0.$$

The second equality follows from that h_n is predictable, and the third equality is due to that S_n^* is a martingale. It is useful to realize that for each n , the process

$$X_n(t) = \sum_{u=1}^t h_n(u) \Delta S_n^*(u)$$

is also a martingale, as a result of the transform from the martingale $\{S_n^*(t)\}$ via the predictable process h_n .

Proof of necessity “ \implies ”

A *contingent claim* is a random variable Y that represents the payoff at time T from a seller (short position) to a buyer (long position). Recall that the sample space $\Omega = \{\omega_1, \dots, \omega_K\}$. Hence the set of possible values $Y(\omega_1), \dots, Y(\omega_K)$ of a contingent claim Y can be considered as an element in \mathbb{R}^K . Let

$$\mathcal{G} = \{Y \in \mathbb{R}^K, Y = G^*(T) \text{ for some trading strategy } h\};$$

$$\mathcal{A} = \{Y \in \mathbb{R}^K, Y \geq 0 \text{ and } Y(\omega) > 0 \text{ for some } \omega \in \Omega\};$$

and

$$\mathcal{G}^\perp = \{Z \in \mathbb{R}^K, Y \cdot Z = 0 \forall Y \in \mathcal{G}\}.$$

Note that \mathcal{G} is a linear subspace of \mathbb{R}^K (why?), and \mathcal{G}^\perp is its orthogonal complement. \mathcal{A} is the (closed) first quadrant (excluding the origin). No arbitrage implies $\mathcal{G} \cap \mathcal{A} = \emptyset$. Furthermore, let

$$\mathcal{W} = \{Y \in \mathbb{R}^K, Y \geq 0, Y_1 + \dots + Y_K = 1\},$$

which is a closed convex subset of \mathcal{A} . It follows from the *Separating Hyperplane Theorem* that there exists $\lambda \in \mathcal{G}^\perp$ such that $\lambda \cdot Y > 0$ for all $Y \in \mathcal{W}$. (See Pliska's book p14 for further detail.) This implies $\lambda(\omega) > 0$ for all $\omega \in \Omega$. Define a probability measure

$$Q(\omega) = \frac{\lambda(\omega)}{\sum_{\omega'} \lambda(\omega')}, \quad \omega \in \Omega.$$

It follows from $Q \in \mathcal{G}^\perp$ that for any predictable process h ,

$$E_Q \left[\sum_{n=1}^N \sum_{t=1}^T h_n(t) \Delta S_n^*(t) \right] = 0.$$

Hence for every n and any predictable process h_n ,

$$E_Q \left[\sum_{t=1}^T h_n(t) \Delta S_n^*(t) \right] = 0.$$

This implies that every S_n^* is a Q -martingale (why?).

Notes:

- (a) The above λ is called a state price vector. More on this later.
- (b) Q is called an EMM because Q is equivalent to P , i.e. for every $\omega \in \Omega$, $Q(\omega) > 0$ if and only if $P(\omega) > 0$.

3.2 Risk neutral valuation of contingent claims and market completeness

A contingent claim Y introduced in Section 3.1 is a contract between a seller and a buyer. Since the seller promises to pay the buyer the amount Y at time T , the buyer normally pays some money to the seller at a certain time $t < T$, when they make the agreement.

Q1: What is the appropriate time t value of this contingent claim Y ? Is it well-defined?

Assume no arbitrage. A contingent claim Y is said to be *marketable* or *attainable* if there exists a self-financing trading strategy h whose value at T satisfies $V(T) = Y$. In this case, h is said to *replicate* or *generate* Y .

Q2: Under what conditions on the market, every contingent claim is marketable?

The next two subsections answer **Q1** and **Q2** respectively.

3.2.1 Law of one price and risk neutral valuation principle

The *law of one price* is said to hold if there do not exist two trading strategies, say h and h' with corresponding value processes denoted by $\{V(t)\}$ and $\{V'(t)\}$, such that $V(T) = V'(T)$ but $V(t) \neq V'(t)$ for some $t < T$. In other words, if the law of one price holds, then there is no ambiguity about the time t value of any marketable claim at any time t .

Proposition 3.1 *No arbitrage \implies the law of one price holds.*

Proof By Theorem 3.1, there is an EMM Q such that all discounted price processes S_n^* , $n = 1, \dots, N$, thus the discounted value process $\{V^*(t)\}$, are Q -martingales. Hence Proposition 3.1 follows (why?).

The converse of Proposition 3.1 is not necessarily true.

Example 3.3 Revisit Example 3.2. With $r = 0.07$, the equation (2.13) yields $q = 1$. In this case, there is a degenerate probability measure Q defined on Ω with $Q(\omega_1) = 1$ and $Q(\omega_k) = 0$ for all $k \neq 1$. Note that Q is not an EMM. But we can still use the equation (2.14) to obtain all values. More generally, the law of one price remains true (why?).

Exercise 3.1 Construct another counterexample in a single period model ($T = 1$).

The following principle is the basis for asset pricing.

Risk neutral valuation principle: Assuming no arbitrage, the time t value of a marketable contingent claim Y is equal to $V(t)$, the time t value of the portfolio that replicates Y . Moreover,

$$V^*(t) = E_Q [Y/B(T) \mid \mathcal{F}_t], \quad t = 0, 1, \dots, T \quad (3.1)$$

for any EMM Q .

Exercise 3.2 Justify this principle.

3.2.2 Complete markets

The example in Lecture 2 illustrates that for a given contingent claim Y , its marketability can be checked by solving a system of linear equations, step by step backwards. Such a tedious procedure is worthwhile because it yields a replicating portfolio when Y is marketable.

Instead of dealing with each individual claim, an alternative approach is to define *complete markets*: a market is said to be complete if every claim in the market is attainable. A general criterion is:

Theorem 3.2 (*Second Fundamental Theorem of Asset Pricing*) *An arbitrage-free market is complete \iff there is a unique EMM Q .*

Proof

“ \implies ” Assuming completeness, every contingent claim Y satisfies $Y = V(T)$ for some self-financing strategy h . Suppose Q_1 and Q_2 are two EMMs with the corresponding expectations denoted by $E_1(\cdot)$ and $E_2(\cdot)$.

$$E_1 [Y/B(T)] = E_1 V^*(T) = E_1 V^*(0) = V^*(0), \quad (3.2)$$

where the second equality is due to that $\{V^*(t)\}$ is a Q_1 -martingale, and the last equality follows from $\mathcal{F}_0 = \{\emptyset, \Omega\}$. By the same token,

$$E_2 [Y/B(T)] = V^*(0). \quad (3.3)$$

Hence $E_1 [Y/B(T)] = E_2 [Y/B(T)]$. This implies $Q_1 = Q_2$ since Y is arbitrary.

“ \impliedby ” Assume the market is arbitrage-free but incomplete, and let \mathcal{C} be the set of all marketable contingent claims. Note that \mathcal{C} is a linear subspace of \mathbb{R}^K . Thus there exists a contingent claim $Y' \in \mathcal{C}^\perp$, with respect to the inner product $(X, Y) = E_Q (XY)$ on \mathbb{R}^K where Q is an EMM. Define

$$Q'(\omega) = \left[1 + \frac{Y'(\omega)}{2 \sup_{\omega \in \Omega} |Y'(\omega)|} \right] Q(\omega), \quad \omega \in \Omega. \quad (3.4)$$

Then

- (i) Q' is a probability measure since $E_{Q'} Y' = 0$;
- (ii) $Q'(\omega) > 0 \quad \forall \omega$ and $Q' \neq Q$;
- (iii) Q' is an EMM because for every n and any predictable process h_n ,

$$E_{Q'} \left[\sum_{t=1}^T h_n(t) \Delta S_n^*(t) \right] = 0.$$

Exercise 3.3 Construct an example of arbitrage-free but incomplete single period model.