Lecture 4 From Binomial Trees to the Black-Scholes Option Pricing Formulas

In this lecture, we will extend the example in Lecture 2 to a general setting of binomial trees, as an important model for a single risky security. It has been extensively used by practitioners in pricing various kinds of derivatives of stocks or bonds. Historically, the model was proposed independently by Cox/Ross/Rubinstein (1979, J. Fin. Econ. **7**, 229-263) and Rendleman/Bartter (1979, J. Fin. **34**, 1093-1110), although it was often referred to as the CRR model. Furthermore, we will show that the celebrated Black-Scholes formulas in option pricing can be derived from the binomial option pricing formulas through an asymptotic argument, provided the parameters in the binomial model are set appropriately.

4.1 The basic binomial tree model

The evolution of a risky security, say stock, is represented by $S = \{S(t),\ t = 0, 1, \dots, T\}$. Starting from an initial (positive) price $S(0)$, assume in each time period the stock price either goes up by a factor $u > 1$ with probability p, or goes down by a factor $0 < d < 1$ with probability $1 - p$. The moves over time are iid Bernoulli random variables. For each t, $S(t) = S(0)u^{n_t}d^{t-n_t}$, where n_t represents the number of up moves up to t.

The bank account process B is deterministic with $B(0) = 1$ and a constant interest rate $0 < r < 1$. Hence $B(t) = (1 + r)^t$.

The filtration $\mathbb{F}$ is taken as the one generated by the history of S. The sample space Ω contains $K = 2^T$ different paths. The underlying probability P is defined by $P(\omega) = p^{U(\omega)}(1 - p)^{T-U(\omega)}$, where $U(\omega)$ represents the total number of up moves in the path ω. We assume $0 < p < 1$ so that $P(\omega) > 0 \quad \forall\omega \in \Omega$.

As for EMMs, we have the following

Proposition 4.1 *There exists a unique EMM $Q \iff d < 1 + r < u$. In this case,*

$$Q(\omega) = q^{U(\omega)}(1 - q)^{T-U(\omega)}, \quad \textit{with} \ \ q = \frac{1 + r - d}{u - d}. \tag{4.1}$$

Proof Let $\xi_t = n_t - n_{t-1}$. Then for every t, $S^*(t) = S^*(t - 1)\ (1 + r)^{-1}u^{\xi_t}d^{1-\xi_t}$. Therefore,

$$\begin{aligned}
& E_Q\ [S^*(t) \mid \mathcal{F}_{t-1}] = S^*(t - 1) \\
\iff \quad & u\ Q(\xi_t = 1 \mid n_{t-1}) + d\ [1 - Q(\xi_t = 1 \mid n_{t-1})] = 1 + r \\
\iff \quad & Q(\xi_t = 1 \mid n_{t-1}) = \frac{1 + r - d}{u - d},
\end{aligned}$$

where $Q(\xi_t = 1 \mid n_{t-1})$ denotes the conditional probability (under Q) that the next move is up given n_{t-1} up moves up to time $t-1$. We can denote this (constant) conditional probability by q since it does not depend on t or n_{t-1}. This implies that $\xi_1, \ldots, \xi_T$ are iid Bernoulli random variables, and the martingale measure Q is given by (4.1). Note that $0 < Q(\omega) < 1$ for every ω if and only if $0 < q < 1$ if and only if $d < 1 + r < u$. The above argument also shows such an EMM Q is unique.

Corollary 4.1 *The binomial tree model is a complete market.*

Exercise 4.1 Show in the binomial tree model, the return $R(t) = \Delta S(t)/S(t-1)$ has the (risk neutral) expectation $E_Q R(t) = r$ (interest rate for the money market) for all $t \geq 1$. This is true in general (later).

4.2 Option pricing using binomial trees

A European option is a contingent claim such that the owner of the option may choose (but with no obligation) to exercise it at an *expiry or expiration time* T and receive the payment Y from the writer of the option. Naturally, the option should be exercised if and only if the payment is positive.

In the simplest case, the contingent claim is expressed as $Y = g(S(T))$ with some function g. Using (3.1) in the binomial tree model, the pricing formula for a European option at time $t = 0, 1, \ldots, T-1$ is given by

$$V(t) = \frac{1}{(1+r)^{T-t}} \sum_{k=0}^{T-t} \binom{T-t}{k} q^k (1-q)^{T-t-k} \; g(S(t) u^k d^{T-t-k}). \tag{4.2}$$

Here are some examples.

Example 4.1 *Call options.* $g(S(T)) = (S(T) - c)^+$ where $c > 0$ is called the *exercise price* or *strike price.* A special case was given in Lecture 2. Note that $S(t) u^k d^{T-t-k} - c > 0 \iff k > \frac{\log\,(c/(S(t)d^{T-t}))}{\log\,(u/d)}$. Let k^* be the smallest k such that this inequality holds. If $k^* > T - t$, then $V(t) = 0$. If $k^* \leq T - t$, then (4.2) becomes

$$V(t) = S(t) \sum_{k=k^*}^{T-t} \binom{T-t}{k} b^k (1-b)^{T-t-k} - \frac{c}{(1+r)^{T-t}} \sum_{k=k^*}^{T-t} \binom{T-t}{k} q^k (1-q)^{T-t-k}, \tag{4.3}$$

where $b = qu/(1+r) \in (0,1)$ (why?). The nice thing about this formula is that it involves two sums of $T - t - k^* + 1$ binomial probabilities.

Example 4.2 *Put options.* Set $g(S(T)) = (c - S(T))^+$. The owner of this option normally chooses to sell the stock at T for the strike price c if $S(T) < c$ (thus make the profit $c - S(T)$), or chooses not to exercise the option if $S(T) \geq c$. A pricing formula similar to (4.3) can be derived easily.

Note: Denote by c_t and p_t respectively, the time t values of the European call and put options with the same expiry T and exercise price c. Since $(S(T)-c)^+ - (c - S(T))^+ = S(T) - c$, we have the following put-call parity

$$c_t - p_t = S(t) - \frac{c}{(1+r)^{T-t}}. \tag{4.4}$$

Example 4.3 *Chooser options.* A chooser option is an agreement that the owner of the option has the right to choose at a fixed *decision time* $T_0 < T$ whether the option is to be a call or a put with a common exercise price c and remaining time to expiry $T - T_0$. To determine the time t value of the chooser option ($t \leq T_0$), notice that the payoff at T is

$$(S(T)-c)^+ I_A + (c - S(T))^+ I_{A^c} = (c - S(T))^+ + I_A\ (S(T) - c),$$

where the event $A = \{c_{T_0} > p_{T_0}\}$, A^c is the complement of A, and I_A is the indicator of A. By the put-call parity, $c_{T_0} - p_{T_0} = S(T_0) - c\ (1+r)^{-(T-T_0)}$, which leads to $A = \{S(T_0) > c\ (1+r)^{-(T-T_0)}\}$. Therefore, the time T_0 value of the chooser option is given by

$$\begin{aligned}
& (1+r)^{-(T-T_0)} E_Q\ [(c - S(T))^+ + I_A\ (S(T) - c) \mid \mathcal{F}_{T_0}] \\
= \ & p_{T_0} + I_A\ E_Q \left[\frac{S(T) - c}{(1+r)^{T-T_0}} \,\middle|\, \mathcal{F}_{T_0} \right] \\
= \ & p_{T_0} + I_A \left[S(T_0) - \frac{c}{(1+r)^{T-T_0}} \right] \\
= \ & p_{T_0} + \left[S(T_0) - \frac{c}{(1+r)^{T-T_0}} \right]^+ .
\end{aligned}$$

Introducing the notation $C(t, T, c)$ (resp. $P(t, T, c)$) for the time t value of a call (resp. put) option with the expiry T and exercise price c, then for any $t = 0, 1, \ldots, T_0$, the time t value $V_{ch}(t)$ of the chooser option can be represented as

$$V_{ch}(t) = P(t, T, c) + C\left(t, T_0,\ c\ (1+r)^{-(T-T_0)}\right), \tag{4.5}$$

or equivalently (why?) as

$$V_{ch}(t) = C(t, T, c) + P\left(t, T_0,\ c\ (1+r)^{-(T-T_0)}\right). \tag{4.6}$$

Exercise 4.2 Verify (4.5) and (4.6).

4.3 The Black-Scholes Option Pricing Formulas

Fix $T > 0$, a real number. For a positive integer n, partition the interval $[0, T)$ into $[(j-1)T/n,\ jT/n)$, $j = 1, \ldots, n$. The previous notation $S(j)$ in the binomial model now represents the stock price at time jT/n. Similarly, $B(j)$ represents the bank account at time jT/n. Let $r_n = rT/n$ be the interest rate, where $r > 0$ is thought of as the instantaneous rate with the continuous compounding, since $\lim_{n\to\infty}(1 + r_n)^n = e^{rT}$. Let $a_n = \sigma\sqrt{\frac{T}{n}}$ where $\sigma > 0$ is interpreted as the *instantaneous volatility.* Set the up and down factors by $u_n = e^{a_n}(1 + r_n)$ and $d_n = e^{-a_n}(1 + r_n)$. Note that $d_n < 1$ for sufficiently large n.

The risk neutral probability, as $n \to \infty$, has the asymptotic expression

$$\begin{aligned} q_n &= \frac{1 + r_n - d_n}{u_n - d_n} \\ &= \frac{1 - e^{-a_n}}{e^{a_n} - e^{-a_n}} \\ &= \frac{a_n - \frac{1}{2}\, a_n^2 + o(a_n^2)}{2a_n + \frac{1}{3}a_n^3 + o(a_n^3)} \\ &= \frac{1}{2} - \frac{1}{4}\, a_n + o(a_n), \end{aligned}$$

where the notation $o(\epsilon)$ with $\epsilon > 0$ means $o(\epsilon)/\epsilon \to 0$ as $\epsilon \to 0$.

Recall the iid Bernoulli random variables ξ_j, $j = 1, \ldots, n$ introduced in Section 4.1, with $Q(\xi_j = 1) = q_n$. The stock price at T is represented as

$$S(n) = S(0)\ u_n^{\xi_1+\cdots+\xi_n}\ d_n^{n-(\xi_1+\cdots+\xi_n)}.$$

Hence the value of the put option at time 0 is given by

$$p_0^{(n)} = (1 + r_n)^{-n}\ E_Q\ (c - S(n))^+ =\ E_Q\ \left(\frac{c}{(1 + r_n)^n} - S(0)\ e^{Y_n}\right)^+, \tag{4.7}$$

where

$$Y_n = \sum_{j=1}^{n}\ Y_{n,j} = \sum_{j=1}^{n}\ \log\left(\frac{u_n^{\xi_j} d_n^{1-\xi_j}}{1 + r_n}\right). \tag{4.8}$$

Note that for fixed n, $Y_{n,1}, \ldots, Y_{n,n}$ are iid random variables with

$$E_Q\ Y_{n,j} = q_n\ \log\frac{u_n}{1 + r_n}\ +\ (1 - q_n)\ \log\frac{d_n}{1 + r_n} = \frac{-1}{2}a_n^2 + o(a_n^2), \tag{4.9}$$

$$E_Q\ Y_{n,j}^2 = a_n^2, \tag{4.10}$$

and

$$E_Q \ |Y_{n,j}|^m = o(a_n^2) \quad \forall \ m = 3, 4, \dots . \tag{4.11}$$

Using characteristic functions [see the note after (4.14)], it follows that Y_n converges in distribution to $N(-\sigma^2 T/2, \ \sigma^2 T)$ as $n \to \infty$. It is noteworthy that the family $\{Y_{n,j}\}$ is a triangular array, hence the asymptotic distribution of Y_n need not always belong to the Gaussian distribution family. In other words, the argument here goes somewhat beyond the basic form of "Central Limit Theorem".

Since

$$|p_0^{(n)} - E_Q \ \left(c \ e^{-rT} - S(0) \ e^{Y_n}\right)^+ | \le c \ |(1+r_n)^{-n} - e^{-rT}|, \quad \text{(why?)} \tag{4.12}$$

we have

$$\begin{aligned}
& \lim_{n\to\infty} p_0^{(n)} \\
= \ & \lim_{n\to\infty} E_Q \ \left(c \ e^{-rT} - S(0) \ e^{Y_n}\right)^+ \\
= \ & \int_{-\infty}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \left[c \ e^{-rT} - S(0) \ \exp\left(-\frac{\sigma^2 T}{2} + \sigma\sqrt{T} z\right)\right]^+ \ dz \\
= \ & c \ e^{-rT} \Phi(-v_2) - S(0) \ \Phi(-v_1),
\end{aligned}$$

where $v_1 = \frac{\log(S(0)/c) + (r+\sigma^2/2) \ T}{\sigma\sqrt{T}}$, $\quad v_2 = v_1 - \sigma\sqrt{T} = \frac{\log(S(0)/c)+(r-\sigma^2/2) \ T}{\sigma\sqrt{T}}$, and Φ is the cumulative distribution function of $N(0,1)$.

This is the Black-Scholes pricing formula for a European put option. We choose to consider put options first since their payoff (or loss) functions are bounded which make the asymptotic argument easier. The following pricing formula for a call option can be derived using put-call parity:

$$\lim_{n\to\infty} c_0^{(n)} = S(0) \ \Phi(v_1) - c \ e^{-rT} \Phi(v_2).$$

Furthermore, by changing 0 to any $t \in (0, T)$ and T to $T - t$, the same argument goes through, which provides the Black-Scholes formulas for pricing the time t value $C(t, T)$ of a (European) call option:

$$C(t,T) = S(t) \ \Phi(v_1) - c \ e^{-r(T-t)} \Phi(v_2), \tag{4.13}$$

and the time t value $P(t, T)$ of a (European) put option:

$$P(t,T) = c \ e^{-r(T-t)} \Phi(-v_2) - S(t) \ \Phi(-v_1), \tag{4.14}$$

where $v_1 = \frac{\log(S(t)/c)+(r+\sigma^2/2) \ (T-t)}{\sigma\sqrt{T-t}}$ and $v_2 = v_1 - \sigma\sqrt{T-t} = \frac{\log(S(t)/c)+(r-\sigma^2/2) \ (T-t)}{\sigma\sqrt{T-t}}$.

Note: To verify that Y_n converges in distribution to $N(-\sigma^2 T/2,\ \sigma^2 T)$ as $n \to \infty$, consider the characteristic function $E_Q\ e^{iwY_n}$ of Y_n where $w \in I\!R$ and $i = \sqrt{-1}$ (imaginary unit in complex analysis). Following the fact that $Y_{n,1}, \ldots, Y_{n,n}$ are iid, and (4.9) — (4.11), we have the Taylor expansion

$$\begin{aligned} E_Q\ e^{iwY_n} &= \prod_{j=1}^{n} E_Q\ e^{iwY_{n,j}} \\ &= \left(1 + iwE_Q Y_{n,j} - \frac{w^2}{2} E_Q Y_{n,j}^2 - \frac{i\theta^3}{3!} E_Q Y_{n,j}^3 \right)^n \\ &\longrightarrow \exp\left(-iw\sigma^2 T/2 - w^2\sigma^2 T/2\right) \end{aligned}$$

as $n \to \infty$, where θ satisfies $|\theta| \leq |w|$. Note that $\exp\left(-iw\sigma^2 T/2 - w^2\sigma^2 T/2\right)$ is just the characteristic function of $N(-\sigma^2 T/2,\ \sigma^2 T)$.

Exercise 4.3 Derive the formula (4.13).