Lecture 6 Cash Flows, Forward Contracts and Futures

We like to broaden our studies in arbitrage pricing from call and put options to other kinds of derivative securities in order to suit the need for quantitative analysis of the evergrowing financial markets. Cash flows, forward contracts, futures and their valuation will be considered in this lecture. Some previous concepts and results require modifications although the basic principle still holds.

6.1 Dividends and returns

Assume the basic set-up in Lecture 2. A *cash flow* can be thought of as a contract with a value $V(t)$ at time t, in which one party of the contract will receive cash payments from the other party on certain future dates. A good example of cash flows is the dividends associated with a stock that we mentioned in Section 5.3.

For $n = 1, \ldots, N$ and $t = 0, 1, \ldots, T$, let $D_n(t)$ be the dividend per unit of security n issued at time t, in particular $D_n(0) = 0$. Let $S_n(t)$ represent the *ex-dividend* price of security n, i.e. the price right after any time t dividend payment. Assume the dividend process is adapted. To check whether arbitrage opportunities exist, it is better to look at *returns* rather than security prices.

Note that a holder of one unit of security n at time $t-1$ will earn a profit $\Delta S_n(t) + D_n(t)$ over the period $(t-1, t]$. The return process $R_n = \{R_n(t)\}$ is defined as follows: $R_n(0) = 0$, and for $t = 1, \ldots, T$,

$$R_n(t) = \begin{cases} \frac{\Delta S_n(t) + D_n(t)}{S_n(t-1)}, & \text{if } S_n(t-1) > 0 \\ 0, & \text{if } S_n(t-1) = 0. \end{cases} \tag{6.1}$$

In particular, $R_0(t) = r(t)$ for the bank account. Moreover, the return process for S_n^* is defined by $R_n^*(0) = 0$, and for $t = 1, \ldots, T$ with $\Delta S_n^*(t) = S_n^*(t) - S_n^*(t-1)$,

$$R_n^*(t) = \begin{cases} \frac{\Delta S_n^*(t) + D_n(t)/B(t)}{S_n^*(t-1)}, & \text{if } S_n^*(t-1) > 0 \\ 0, & \text{if } S_n^*(t-1) = 0. \end{cases} \tag{6.2}$$

Also, $R_0^*(t) \equiv 0$ for the bank account.

If $X = \{X(t),\ t = 0, 1, \ldots, T\}$ is a martingale under a probability measure Q and with respect to $\mathbb{F}$, then the increment process $\Delta X = \{\Delta X(t)\}$ is called the corresponding *martingale difference sequence*, i.e.

$$E_Q\left(\Delta X(t) \mid \mathcal{F}_{t-1}\right) = 0 \quad \forall\, t = 1, \ldots, T.$$

Here is a modification of Theorem 3.1 in the case of dividend-paying securities.

Theorem 6.1 *No arbitrage* $\Longleftrightarrow$ *there is a probability measure* Q *with* $Q(\omega) > 0\ \forall \omega \in \Omega$*, such that* $R_n^* = \{R_n^*(t),\ t = 1, \ldots, T\}$ *is a* Q*-martingale difference sequence,* $n = 1, \ldots, N$*. We still call* Q *an EMM.*

The proof of Theorem 3.1 still applies if we replace $\Delta S_n(t)$ there by $\Delta S_n(t) + D_n(t)$.

6.2 Forward contracts and prices

A *forward contract* is an agreement made between two parties at time t in which the buyer agrees to buy an underlying asset on a certain specified future date τ (with $t < \tau \leq T$) for a *delivery price*; while the seller agrees to sell the asset on the same date τ for the same price. At the maturity date τ, the seller delivers the asset to the buyer in return for a cash payment equal to the delivery price. Although the assets in forward contracts can be commodities, we only focus on securities. The delivery price that would make the time t value of the forward contract to either party zero is called the *forward price*, denoted by $FO(t)$. A basic problem is to determine $FO(t)$ for a forward contract.

Since we only consider those forward contracts that can be replicated by self-financing trading strategies, we assume a complete market without loss of generality. Let $S(t)$ be the time t price of an underlying security, and let $b_{t,\tau} = \{E_Q[B(t)/B(\tau) \mid \mathcal{F}_t]\}^{-1}$, where Q is an EMM, represent the growth factor in the period $(t, \tau]$. Then we have

$$FO(t) = S(t)\ b_{t,\tau}. \tag{6.3}$$

Note: The forward price $FO(t)$ of a forward contract is *not* the time t value of the forward contract. The time t value of the forward contract is zero. Also the general definition of $b_{t,\tau}$ applies to the case with an adapted stochastic interest rate process $\{r(t)\}$, which will be necessary later on when we study fixed-income derivatives.

There are a couple of ways to verify (6.3). One is to use risk neutral valuation. Notice that the time τ value of the forward contract is $S(\tau) - FO(t)$. By the definition of $FO(t)$, we should have $E_Q\{[S(\tau) - FO(t)]\ B(t)/B(\tau) \mid \mathcal{F}_t\} = 0$, hence

$$\begin{aligned} S(t) &= E_Q[S(\tau)\ B(t)/B(\tau) \mid \mathcal{F}_t] \\ &= E_Q[FO(t)\ B(t)/B(\tau) \mid \mathcal{F}_t] = FO(t)\ E_Q[B(t)/B(\tau) \mid \mathcal{F}_t], \end{aligned}$$

which implies (6.3). The last equality is due to that $FO(t)$ is measurable with respect to $\mathcal{F}_t$.

Another way to verify (6.3) is via the standard arbitrage argument. Technically, that would need a zero-coupon bond with maturity τ. Here we only present the argument in the special case of $r(t) \equiv r$ thus $b_{t,\tau} = (1 + r)^{\tau - t}$. This simplification makes the argument more transparent.

Suppose first $FO(t) < S(t)\ b_{t,\tau}$. An investor can take a long position in the forward contract, short the security at time t, and deposit the proceeds in the bank. At time τ, the security is purchased under the term of the forward contract for $FO(t)$, the short position in the security is closed out, and a profit $S(t)\ b_{t,\tau} - FO(t) > 0$ is earned.

On the other hand, suppose $FO(t) > S(t)\ b_{t,\tau}$. Then an investor can borrow the amount $S(t)$ at time t, buy the security, and take a short position in the forward contract. At time τ, the security is sold under the term of the forward contract for $FO(t)$. After using the amount $S(t)\ b_{t,\tau}$ to repay the loan, a profit $FO(t) - S(t)\ b_{t,\tau} > 0$ is realized.

The above result can be generalized to the case of dividend-paying securities. Suppress the subscript n in the notation given in Section 6.1 and denote the dividend process (a cash flow) by $D = \{D(t)\}$. At any fixed time $s > t$, $D(s)$ is simply a contingent claim, thus its time t value is $d_{t,s} = E_Q[D(s)B(t)/B(s) \mid \mathcal{F}_t]$, called the time t *present value* of $D(s)$. Following Theorem 6.1, we have

$$E_Q[\Delta S^*(t+1) + D(t+1)/B(t+1) \mid \mathcal{F}_t] = 0 \quad \forall t = 0, 1, \ldots, T-1; \tag{6.4}$$

and for $\tau > t$,

$$S^*(t) = E_Q\left[\sum_{s=t+1}^{\tau} \frac{D(s)}{B(s)} + S^*(\tau) \;\middle|\; \mathcal{F}_t\right]. \tag{6.5}$$

Hence the time t present value of a cash flow (such as dividends) $D(s),\ t < s \le \tau$ is given by

$$\sum_{s=t+1}^{\tau} d_{t,s} = S(t) - E_Q[S(\tau)B(t)/B(\tau) \mid \mathcal{F}_t]. \tag{6.6}$$

Now we have

Proposition 6.1 *With a security process S and a dividend process D, the time t forward price $FO(t)$ of a forward contract, which is received and paid at time $\tau > t$, is given by*

$$FO(t) = \left[S(t) - \sum_{s=t+1}^{\tau} d_{t,s}\right]\ b_{t,\tau} = E_Q[S(\tau)B(t)/B(\tau) \mid \mathcal{F}_t]\ b_{t,\tau}. \tag{6.7}$$

Proof Once again, we start with the equation

$$E_Q\{[S(\tau) - FO(t)]\ B(t)/B(\tau) \mid \mathcal{F}_t\} = 0 \tag{6.8}$$

that defines $FO(t)$. But the ex-dividend price S does not satisfy

$$E_Q[S(\tau)B(t)/B(\tau) \mid \mathcal{F}_t] = S(t).$$

Instead, it follows from (6.6) that

$$E_Q[S(\tau)B(t)/B(\tau) \mid \mathcal{F}_t] = S(t) - \sum_{s=t+1}^{\tau} d_{t,s},$$

which matches $FO(t)$ after multiplying both sides by $b_{t,\tau}$.

Alternatively, we may consider the following portfolio that replicates $S(\tau)$:

At time t: buy one unit of security for $S(t)$;

Also at time t: for each $s = t+1, \ldots, \tau$, borrow $d_{t,s}$ by undertaking the negative of the strategy that replicates the time s receipt $D(s)$, then at time s use the dividend payment $D(s)$ to settle the liability under the strategy.

The time t value of this replicating portfolio also amounts to

$$E_Q[S(\tau)B(t)/B(\tau) \mid \mathcal{F}_t] = S(t) - \sum_{s=t+1}^{\tau} d_{t,s}.$$

Example 6.1 Consider the binomial model with parameters u, d, r, and a dividend issued with the constant yield λ on the ex-dividend date τ, where $1 \leq \tau \leq T$.

Suppose $S(\tau - 1) > 0$, then (6.1) yields the return

$$R(\tau) = \frac{\Delta S(\tau) + D(\tau)}{S(\tau-1)} = u - 1 \quad \text{or} \quad d - 1, \tag{6.9}$$

which is the same as in the case without dividends. The same calculation reduces (6.2) to

$$R^*(\tau) = \frac{u}{1+r} - 1 \quad \text{or} \quad \frac{d}{1+r} - 1. \tag{6.10}$$

According to Theorem 6.1, we can set

$$E_Q[R^*(\tau)|\mathcal{F}_{\tau-1}] = q\ \left(\frac{u}{1+r} - 1\right) + (1-q)\ \left(\frac{d}{1+r} - 1\right) = 0, \tag{6.11}$$

and obtain that $q = \frac{1+r-d}{u-d}$ which is the risk neutral probability provided $d < 1+r < u$. Therefore, the risk neutral probability q (thus the corresponding EMM Q) remains the same as in the case without dividends.

Furthermore, (6.7) gives us the one-step forward price $FO(\tau - 1)$ for a forward contract with maturity τ:

$$FO(\tau - 1) = S(\tau - 1)\ (1+r)(1-\lambda). \tag{6.12}$$

Any contingent claim Y defined at τ has its time $\tau - 1$ value $V(\tau-1) = (1+r)^{-1}E_Q[Y|\mathcal{F}_{\tau-1}]$. For example, given $S(\tau-1) = \zeta$, the time $\tau - 1$ value of an European call option $Y = [S(\tau) - c]^+$ is given by

$$V(\tau-1) = (1+r)^{-1}\{q\ [u(1-\lambda)\zeta - c]^+ + (1-q)\ [d(1-\lambda)\zeta - c]^+\}. \tag{6.13}$$

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6.3 Futures vs forward contracts

Like a forward contract, a futures contract is an agreement made at time t which involves a cash payment between two parties in a specified future time with a specified delivery price. However, there are a number of noteworthy differences between futures and forward contracts that we summarize as follows. See Hull's book for more details.

- Trading
  - *Forward:* Traded *over-the-counter* (OTC) anywhere anytime by telephone or other communications between individual buyers and sellers;
  - *Futures:* Traded on some centralized exchange floors, such as Chicago Board of Trade (CBOT) and New York Futures Exchange (NYFE), during their business hours.
- Standardization
  - *Forward:* Not standardized — almost everything in a contract is based on private negotiation: the underlying assets to be traded, price, delivery time and location, etc.
  - *Futures:* A standardized contract specifies the underlying asset to be traded, price, delivery date (chosen from among a limited number of dates each year) and location.
- Delivery
  - *Forward:* Delivery or final cash settlement usually take place.
  - *Futures:* A very small proportion of contracts are actually delivered. A majority of contracts are closed out prior to maturity.
- Collateral and margin
  - *Forward:* Collateral level negotiable; no adjustment for daily price fluctuations, settlement only at end of contract. Hence market participants bear the risk of counter-party defaulting.

- ◦ *Futures:* A central clearinghouse is associated with each exchange to regulate the trade. Each investor is required to initiate a *margin* account as a security deposit. Positions in futures contracts are governed by a daily settlement procedure, referred to as *marking to market*. The investor is allowed to withdraw any balance in the margin account in excess of the initial margin. To ensure that the balance in the margin account never becomes negative, a *maintenance margin*, which is somewhat lower than the initial margin, is set. If the balance in the margin account falls below the maintenance margin, the investor receives a *margin call* and is expected to top up the margin account to the initial margin level the next day. The extra funds deposited are called a *variation margin*. A failure to provide the variation margin would result in closing out the investor's position, i.e. a broker associated with the clearinghouse would sell the contract.

- Information and *liquidity* (important to statistical analysis of financial market, sometimes referred to as *empirical finance*)
  - ◦ *Forward:* No volume information available; low liquidity due to a large variety of contract terms.
  - ◦ *Futures:* Volume information published; high liquidity due to standardized contracts.

6.4 Futures prices

Marking to market is a device in futures market to reduce the risk of default inherent in forward contracts. Recall that the value of a forward contract need not equal zero except at the time when the contract starts. In contrast, the value of a futures contract is maintained at zero at all times via marking to market. Thus either party can close out his/her position at any time.

The futures price of a futures contract, specified at the time when the futures contract is entered into, is not like the forward price which is defined by the equation (6.8). Instead, it is determined on a given futures exchange by the usual law of supply and demand, similar to stock prices.

For $m = 1, \dots, M$, let $FU_m(t, t_m)$ be the futures price of a futures contract with starting date t and maturity date t_m, and $\Delta FU_m(s, t_m) = FU_m(s, t_m) - FU_m(s-1, t_m)$, $t < s \le t_m$. An investor taking a long position either receives a payment $\Delta FU_m(s, t_m)$ at time s, or makes a payment $-\Delta FU_m(s, t_m)$ if the futures price falls. These fluctuations are reflected in marking to market. We will study the connections between

(i) the futures price process $FU_m(\cdot, t_m)$ and the underlying security process $S_m(\cdot)$;

(ii) the futures price $FU_m(t, t_m)$ and the corresponding forward price $FO(t)$ with the same maturity t_m.

Let $(h;\ h^f) = (h_0, h_1, \dots, h_N;\ h_1^f, \dots, h_M^f)$ denote the portfolio process in a combined spot/futures market, where all the components are predictable processes. For each m, $h_m^f(t)$ represents the position in futures contract m held from time $t-1$ to t. We assume that $FU_m(t, t_m) = 0$ and $h^f_m(t) = 0$ for any $t > t_m$.

Consider the value process. The time t value of a portfolio right *before* marking to market is

$$V(t) = h_0(t)B(t) + \sum_{n=1}^{N} h_n(t)S_n(t) + \sum_{m=1}^{M} h^f_m(t)\Delta FU_m(t, t_m); \tag{6.14}$$

and the time t value of the portfolio right *after* marking to market is

$$V^+(t) = h_0(t+1)B(t) + \sum_{n=1}^{N} h_n(t+1)S_n(t). \tag{6.15}$$

Compared to (6.14), the absence of terms involving futures prices in (6.15) is due to the updating adjustment via marking to market. A trading strategy $(h;\ h^f)$ is said to be self-financing if its value process satisfies

$$V(t) = V^+(t) \quad \forall\, t = 1, \ldots, T-1. \tag{6.16}$$

Since a futures trader must have positive wealth, we cannot define arbitrage by starting with $V(0) = 0$. Nevertheless, other definitions of arbitrage would work. An arbitrage opportunity in a spot/futures market corresponds to a self-financing strategy $(h;\ h^f)$ whose value satisfies

(a) $V^*(T) \geq V^*(0) \quad \forall \omega$, and

(b) $V^*(T) > V^*(0) \quad$ for some ω.

An alternative to (a) and (b) is

(a$'$) $G^*(T) \geq 0 \quad \forall \omega$, and

(b$'$) $G^*(T) > 0 \quad$ for some ω,

where the discounted gain G^* is specified through its increment in $(t-1, t]$

$$\Delta G^*(t) = \sum_{n=1}^{N} h_n(t)\Delta S^*_n(t) + \sum_{m=1}^{M} h^f_m(t)\Delta FU_m(t, t_m)/B(t), \quad t = 1, \ldots, T. \tag{6.17}$$

The following theorem is an extension of Theorem 3.1 to the case of spot/futures market.

Theorem 6.2 *For a combined spot/futures market, no arbitrage $\iff$ there is an EMM Q, such that for every $n = 1, \ldots, N$, the process S^*_n is a Q-martingale; moreover, for every $m = 1, \ldots M$,*

$$E_Q[\Delta FU_m(t, t_m)/B(t) \mid \mathcal{F}_{t-1}] = 0, \quad t = 1, \ldots, T. \tag{6.18}$$

The proof is left as an exercise, since the argument in proving Theorem 3.1 can be modified straightforwardly.

Corollary 6.1 *Consider the following special cases of Theorem 6.2.*

(a) *Suppose B is a predictable process. Then (6.18) is equivalent to that $FU_m(\cdot, t_m)$ is a Q-martingale for each m.*

(b) *Suppose B is deterministic with $r(t) \equiv r$. Then for every m and t,*

$$FU_m(t, t_m) = FO(t), \tag{6.19}$$

where $FO(t)$ is the forward price of a forward contract starting at t and ending at t_m.

Proof (a) is trivial. For (b), note that $FU_m(t_m, t_m) = S_m(t_m)$, i.e. the futures price converges to the spot price of the underlying security at the delivery date. The martingale properties of $FU_m(\cdot, t_m)$ and $S_m(\cdot)$ imply that

$$\begin{aligned} FU_m(t, t_m) &= E_Q[FU_m(t_m, t_m) \mid \mathcal{F}_t] = E_Q[S_m(t_m) \mid \mathcal{F}_t] \\ &= (1 + r)^{t_m - t} S_m(t) = FO(t). \end{aligned}$$

Note: (6.19) remains valid when underlying securities pay dividends. This just follows from (6.7).

Example 6.2 For a single period market, (6.19) always holds even when B is random, adapted and nonpredictable. This is intuitively obvious since there is no marking to market. The holder of a futures contract can only wait until the delivery date — same as in a similar forward contract. To verify (6.19), note that (6.18) in this case is written as

$$E_Q[\Delta FU_m(1, 1)/B(1) \mid \mathcal{F}_0] = E_Q\{[S_m(1) - FU_m(0, 1)]/B(1) \mid \mathcal{F}_0\} = 0,$$

which is just the equation to define the forward price $FO(0)$.

Example 6.3 Consider a binomial tree model with $T = 2$. Identify $\omega_1 = uu$, $\omega_2 = ud$, $\omega_3 = du$ and $\omega_4 = dd$. Let $S(0) = 1$, $u = 1.07$, $d = 0.92$; $B(0) = B(1) = 1$ and

$$B(2) = \begin{cases} 1.03, & \text{on } \{\omega_1, \omega_2\} \\ 1.04, & \text{on } \{\omega_3, \omega_4\} \end{cases}$$

Since there is only a single stock, we suppress the subscript m, and calculate Q, $FU(0, 2)$ and $FO(0)$ (also with delivery date $T = 2$).

- To calculate Q, write $E_Q[S(t)/B(t) \mid \mathcal{F}_{t-1}] = S(t-1)/B(t-1)$:

  For $t = 2$:

  $1.07q_1 + 0.92(1 - q_1) = 1.03$ thus $q_1 = \frac{11}{15}$ on $\{\omega_1, \omega_2\}$;

  $1.07q_2 + 0.92(1 - q_2) = 1.04$ thus $q_2 = \frac{12}{15}$ on $\{\omega_3, \omega_4\}$.

  For $t = 1$:

  $q_0 = \frac{1-d}{u-d} = \frac{8}{15}$.

  Hence $Q(\omega_i)$, $i = 1, 2, 3, 4$, are equal to $q_0 q_1$, $q_0(1 - q_1)$, $(1 - q_0)q_2$ and $(1 - q_0)(1 - q_2)$ respectively.

- (6.18) is carried out in two steps.

  For $t = 2$, $S(1)/B(1) = FU(1, 2)\ E_Q[B^{-1}(2) \mid \mathcal{F}_1]$, i.e.

  $FU(1, 2) = 1.07 \cdot 1.03 = 1.1021$ on $\{\omega_1, \omega_2\}$;

  $FU(1, 2) = 0.92 \cdot 1.04 = 0.9568$ on $\{\omega_3, \omega_4\}$.

  For $t = 1$,

  $FU(0, 2) = E_Q FU(1, 2) = 1.1021q_0 + 0.9568(1 - q_0) = \frac{155144}{150000} \approx 1.03429$.

- (6.3) yields

  $FO(0) = S(0)\ [E_Q B^{-1}(2)]^{-1} = [1.03^{-1}q_0 + 1.04^{-1}(1 - q_0)]^{-1} = \frac{160680}{155300} \approx 1.03464$.

In this example, B is random and predictable, but $FU(0, 2) \neq FO(0)$.

6.5 Options on futures

Options or more general contingent claims can be defined as derivatives of underlying securities *and/or* futures in a combined spot/futures market. Risk neutral valuation based on Theorem 6.2 can be carried out. Any marketable contingent claim can be hedged via replicating portfolios $(h;\ h^f)$. The only technical subtlety to keep in mind is that when using an EMM Q, the *discounted* security price S^* is a Q-martingale while the *undiscounted* futures price $FU(\cdot, \tau)$ is a Q-martingale (assuming B is predictable). Hence the price of a contingent claim depends on whether it is defined as a derivative on a security or on a futures contract. Some related problems will be given in homework assignments.