

Lecture 7 Introduction to Fixed-income Market

The *fixed-income* market is an important part of the global financial market in which various interest rate securities, such as bonds and their derivatives, are traded. The total volume of fixed-income securities traded in the market is much greater than that of equities such as common stocks. Mathematical models for fixed-income derivatives are also more subtle and complex. We will introduce some basic concepts, models and derivatives in the next few lectures.

7.1 Zero-coupon bonds, yields and forward rates

Assume the basic setting in Section 2.1 with a sample space Ω , an underlying probability measure P , a filtration \mathbb{F} , a finite horizon T , and with the following extension: the bank account B is assumed to be random and predictable with interest rate $r(t) > 0$ for all $t = 1, \dots, T$. This means that the interest rate $r(t)$ for borrowing or lending over the period $(t-1, t]$ is known at time $t-1$. The process B is usually taken as a *numéraire*, i.e. the unit of an account in which other assets are denominated. $r(t)$ is called the *spot* rate or *short* rate.

- Various bonds are considered as risky securities. One of them is a collection of *zero-coupon* or *discount* bonds, denoted by $\{B(t, \tau) : 0 \leq t \leq \tau, \tau = 1, \dots, T\}$, where $B(t, \tau)$ represents the time t price of a zero-coupon bond with maturity τ . Sometimes we simply call a zero coupon bond with maturity τ a τ -*bond*. Assume that for each τ , the process $B(\cdot, \tau)$ is positive and adapted to \mathbb{F} , in particular, $B(\tau, \tau) = 1$ at par (the nominal value is \$1 at maturity). On the other hand, for each t , the collection $\{B(t, \tau), t < \tau \leq T\}$ is called the time t *term structure* of zero-coupon bond prices. Hence we are dealing with a process $B(\cdot, \cdot)$ with double indices, which makes the analysis considerably more challenging.

Now we consider a couple of other term structures equivalent to $B(t, \cdot)$, and we assume $t \leq \tau$.

- Let $Y(t, \tau)$ be the *constant* interest rate (or return) at which $B(t, \tau)$, when compounded during $(t, \tau]$, would reach \$1 at time τ , called the *yield to maturity*, i.e.

$$B(t, \tau) [1 + Y(t, \tau)]^{\tau-t} = 1, \tag{7.1}$$

or equivalently,

$$Y(t, \tau) = [B(t, \tau)]^{\frac{-1}{\tau-t}} - 1. \tag{7.2}$$

In particular, $Y(t-1, t) = r(t)$, the spot rate at one period before maturity. For each t , the collection $\{Y(t, \tau), t < \tau \leq T\}$ is called the time t term structure of interest rates or *yield curve*. The two term structures $B(t, \cdot)$ and $Y(t, \cdot)$ are equivalent.

- Let $f(t, \tau)$ be the “short rate” such that
 - (i) it is locked into at time t ;
 - (ii) it is applied to the period $(\tau, \tau + 1]$;
 - (iii) it is associated with a $(\tau + 1)$ -bond.

Then we have

$$\frac{B(t, \tau + 1)}{B(t, \tau)} [1 + f(t, \tau)] = 1, \quad (7.3)$$

or equivalently,

$$f(t, \tau) = \frac{B(t, \tau)}{B(t, \tau + 1)} - 1. \quad (7.4)$$

To see why this is true, assume there exists an EMM Q such that for every τ , the discounted τ -bond process $B^*(\cdot, \tau)$ is a Q -martingale, i.e.

$$B(t - 1, \tau) = E_Q[B(t, \tau)B(t - 1)/B(t) \mid \mathcal{F}_{t-1}], \quad 1 \leq t \leq \tau. \quad (7.5)$$

Notice that in a fixed-income market, zero coupon bonds play the same role as risky securities like common stocks in a stock market. We can consider a forward contract on a $(\tau + 1)$ -bond, delivered at τ . According to (6.3), its forward price $FO(t)$ at time t is given by

$$FO(t) = B(t, \tau + 1) b_{t, \tau} = \frac{B(t, \tau + 1)}{E_Q[B(t)/B(\tau) \mid \mathcal{F}_t]} = \frac{B(t, \tau + 1)}{B(t, \tau)}, \quad (7.6)$$

where the last equality is due to that

$$B(t, \tau) = E_Q[B(t)/B(\tau) \mid \mathcal{F}_t]. \quad (7.7)$$

In particular,

$$f(t, t) = r(t + 1). \quad (7.8)$$

Exercise 7.1 Verify the following results:

$$B(t, \tau) = E_Q\{[(1 + r(t + 1)) \cdots (1 + r(\tau))]^{-1} \mid \mathcal{F}_t\}; \quad (7.9)$$

$$r(t) = B^{-1}(t - 1, t) - 1; \quad (7.10)$$

$$B(t, \tau) = \prod_{s=t+1}^{\tau} [1 + f(t, s - 1)]^{-1}; \quad (7.11)$$

For every t , $B(t, \tau)$ is strictly decreasing in τ .

7.2 Examples of term structure models

There are two different kinds of term structure models: *spot rate models* (*short rate models*, *equilibrium models*, etc.), in which initial term structures are outputs from the models; and *yield curve models* (*forward rate models*, *no arbitrage models*, etc.), in which initial term structures are inputs to the models. Before general studies of these two approaches, we illustrate them in two simple examples respectively.

In both examples, we assume $T = 3$, $\Omega = \{\omega_i, i = 1, 2, 3, 4\}$, and the following partitions that generate the filtration \mathbb{F} :

$$\mathcal{P}_1 = \{\omega_1, \omega_2, \omega_3, \omega_4\};$$

$$\mathcal{P}_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\};$$

and

$$\mathcal{P}_3 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}.$$

Example 7.1

- Start with a spot rate binomial tree, Figure 7.1, in which

$$r(1) = 0.04;$$

$$r(2) = \begin{cases} 0.06 & \text{on } \{\omega_1, \omega_2\}, \\ 0.03 & \text{on } \{\omega_3, \omega_4\}; \end{cases}$$

and

$$r(3) = \begin{cases} 0.07 & \text{on } \{\omega_1\}, \\ 0.05 & \text{on } \{\omega_2\}, \\ 0.04 & \text{on } \{\omega_3\}, \\ 0.02 & \text{on } \{\omega_4\}. \end{cases}$$

- By (7.10) $B(t-1, t) = [1 + r(t)]^{-1}$, we obtain that

$$B(0, 1) = 0.9615;$$

$$B(1, 2) = \begin{cases} 0.9434 & \text{on } \{\omega_1, \omega_2\}, \\ 0.9709 & \text{on } \{\omega_3, \omega_4\}; \end{cases}$$

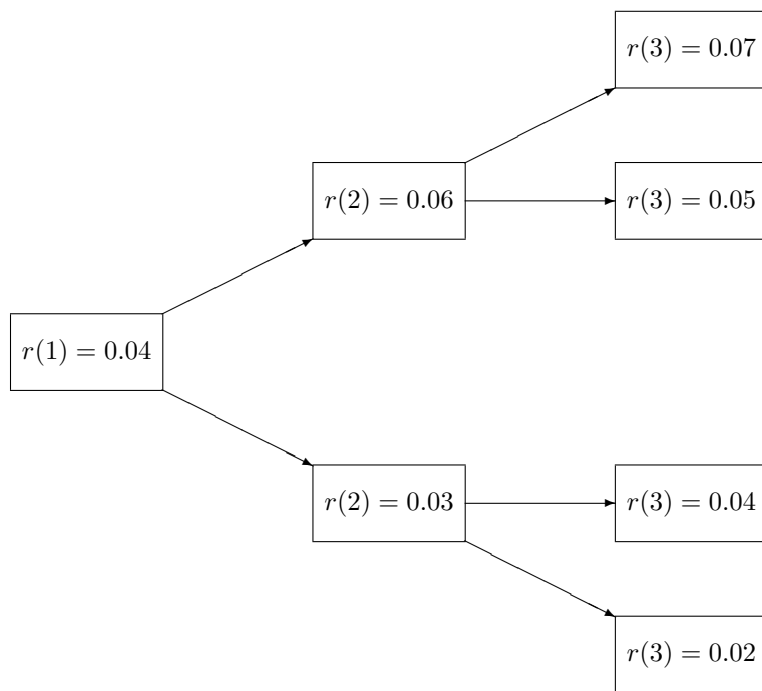


Figure 7.1: Spot rate tree

and

$$B(2, 3) = \begin{cases} 0.9346 & \text{on } \{\omega_1\}, \\ 0.9524 & \text{on } \{\omega_2\}, \\ 0.9615 & \text{on } \{\omega_3\}, \\ 0.9804 & \text{on } \{\omega_4\}. \end{cases}$$

- At this point, we have some flexibility. We choose an EMM Q by assigning $Q(\omega_i) = 1/4$, $i = 1, 2, 3, 4$.
- It follows from (7.7) or (7.9) that

$$B(0, 2) = E_Q[1/B(2)] = \frac{1}{1.04} \cdot \frac{1}{2} \cdot \left(\frac{1}{1.06} + \frac{1}{1.03} \right) = 0.9203,$$

$$\begin{aligned} B(0, 3) &= E_Q[1/B(3)] \\ &= \frac{1}{1.04} \cdot \frac{1}{4} \cdot \left(\frac{1}{1.06 \cdot 1.07} + \frac{1}{1.06 \cdot 1.05} + \frac{1}{1.03 \cdot 1.04} + \frac{1}{1.03 \cdot 1.02} \right) = 0.8811, \end{aligned}$$

and

$$\begin{aligned} B(1, 3) &= E_Q \left\{ \frac{1}{[1+r(2)][1+r(3)]} \mid \mathcal{F}_1 \right\} = \frac{1}{1+r(2)} E_Q \left\{ \frac{1}{1+r(3)} \mid \mathcal{F}_1 \right\} \\ &= \begin{cases} \frac{1}{1.06} \cdot \frac{1}{2} \left(\frac{1}{1.07} + \frac{1}{1.05} \right) = 0.8901 & \text{on } \{\omega_1, \omega_2\}, \\ \frac{1}{1.03} \cdot \frac{1}{2} \left(\frac{1}{1.04} + \frac{1}{1.02} \right) = 0.9427 & \text{on } \{\omega_3, \omega_4\}. \end{cases} \end{aligned}$$

The result is summarized in Table 7.1.

ω	$B(0, 1)$	$B(0, 2)$	$B(1, 2)$	$B(0, 3)$	$B(1, 3)$	$B(2, 3)$	Q
ω_1	0.9615	0.9203	0.9434	0.8811	0.8901	0.9346	0.25
ω_2	~	~	~	~	~	0.9524	~
ω_3	~	~	0.9709	~	0.9427	0.9615	~
ω_4	~	~	~	~	~	0.9804	~

Table 7.1: Data for Example 7.1. The notation \sim represents the same number as above.

A drawback of the approach used in Example 7.1 is that the initial (time 0) term structure is derived from the model, thus need not be consistent with the given data (the initial term structure data is available at time 0). An alternative approach is to model the entire family of yield curves *conditioning* on the given initial term structure.

Example 7.2

- Suppose the initial term structure is given by

$$B(0, 1) = 0.96 \quad [\text{thus } r(1) = 0.0417],$$

$$B(0, 2) = 0.92,$$

and

$$B(0, 3) = 0.88.$$

- If we let $Q(\omega_i) = 1/4$, $i = 1, 2, 3, 4$ as in Example 7.1, there is still some flexibility in specification of $r(2)$ and $r(3)$, or equivalently, $B(t, \cdot)$, $t = 1, 2$. We will move forward one period at a time.
- For $t = 2$, set $r(2) = 0.06$ on $\{\omega_1, \omega_2\}$. Then

$$B(0, 2) = \frac{1}{1 + r(1)} E_Q \left[\frac{1}{1 + r(2)} \right],$$

i.e.

$$0.92 = \frac{1}{1.0417} \cdot \frac{1}{2} \left[\frac{1}{1.06} + \frac{1}{1 + r(2)} \right],$$

which gives rise to

$$r(2) = 0.0274 \quad \text{on } \{\omega_3, \omega_4\}.$$

Hence

$$B(1, 2) = \frac{1}{1 + r(2)} = \begin{cases} 0.9434 & \text{on } \{\omega_1, \omega_2\}, \\ 0.9733 & \text{on } \{\omega_3, \omega_4\}. \end{cases}$$

- For $t = 3$, set

$$r(3) = \begin{cases} 0.07 & \text{on } \{\omega_1\}, \\ 0.05 & \text{on } \{\omega_2\}, \\ 0.04 & \text{on } \{\omega_3\}. \end{cases}$$

Notice the constraint

$$B(0, 3) = \frac{1}{1 + r(1)} E_Q \left\{ \frac{1}{[1 + r(2)][1 + r(3)]} \right\},$$

i.e.

$$0.88 = \frac{1}{1.0417} \cdot \frac{1}{4} \left[\frac{1}{1.06 \cdot 1.07} + \frac{1}{1.06 \cdot 1.05} + \frac{1}{1.0274 \cdot 1.04} + \frac{1}{1.0274 \cdot 1 + r(3)} \right],$$

which implies

$$r(3) = 0.0238 \quad \text{on } \{\omega_4\}.$$

Therefore,

$$B(2, 3) = \frac{1}{1 + r(3)} = \begin{cases} 0.9346 & \text{on } \{\omega_1\}, \\ 0.9524 & \text{on } \{\omega_2\}, \\ 0.9615 & \text{on } \{\omega_3\}, \\ 0.9768 & \text{on } \{\omega_4\}. \end{cases}$$

Finally,

$$\begin{aligned} B(1, 3) &= \frac{1}{1 + r(2)} E_Q \left[\frac{1}{1 + r(3)} \mid \mathcal{F}_1 \right] \\ &= \begin{cases} \frac{1}{1.06} \cdot \frac{1}{2} \left(\frac{1}{1.07} + \frac{1}{1.05} \right) = 0.8901 & \text{on } \{\omega_1, \omega_2\}, \\ \frac{1}{1.0274} \cdot \frac{1}{2} \left(\frac{1}{1.04} + \frac{1}{1.0238} \right) = 0.9433 & \text{on } \{\omega_3, \omega_4\}. \end{cases} \end{aligned}$$

The result is summarized in Table 7.2.

ω	$B(1, 2)$	$B(1, 3)$	$B(2, 3)$	$r(2)$	$r(3)$	Q
ω_1	0.9434	0.8901	0.9346	0.06	0.07	0.25
ω_2	\sim	\sim	0.9524	\sim	0.05	\sim
ω_3	0.9733	0.9433	0.9615	0.0274	0.04	\sim
ω_4	\sim	\sim	0.9768	\sim	0.0238	\sim

Table 7.2: Data for Example 7.2. The notation \sim represents the same number as above.

Notes:

- (a) In Example 7.2, one can first specify all $B(\cdot, \cdot)$ in a consistent manner without using Q and $r(\cdot)$. Then Q and $r(\cdot)$ will be determined accordingly.
- (b) In practice, *trinomial* trees or bushy trees with even more branches from each node are often used to study *multi-factor* term structure models.

7.3 Spot rate Markov chains

We introduce two methods in modeling spot rates: *Markov chains* (MC) and *stochastic difference equations* (SDE). Each method has its own advantage. MC makes use of binomial trees, and SDE is linked to models in continuous-time finance, which we will discuss later. Overall, MC provides somewhat more general models than SDE.

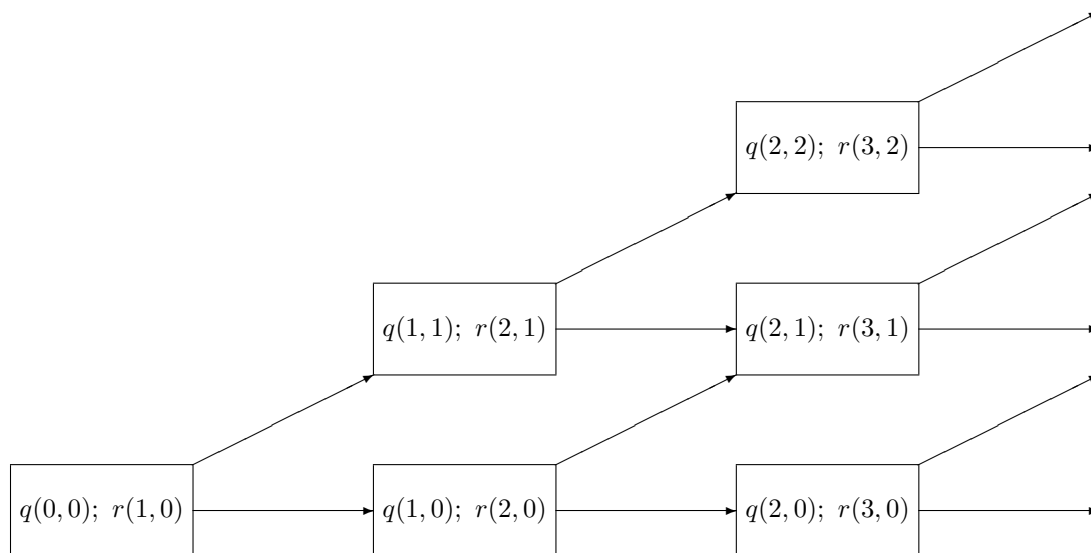


Figure 7.2: Binomial tree of Markov chain X and spot rate r

A stochastic process $X = \{X(t), t = 0, 1, \dots, T\}$ is called a Markov chain (under a probability measure P) with a state space \mathcal{S} if for every $t = 0, 1, \dots, T - 1$,

$$P\{X(t+1) = x_{t+1} \mid X(s) = x_s, 0 \leq s \leq t\} = P\{X(t+1) = x_{t+1} \mid X(t) = x_t\}$$

for all possible values $x_s \in \mathcal{S}$, $0 \leq s \leq t + 1$. Several quantities of interest, such as the spot rate $r(\cdot)$ and the zero-coupon bond $B(\cdot, \cdot)$ can be modelled as Markov chains. A convenient way is to construct an underlying Markov chain X such that $r(\cdot)$, $B(\cdot, \cdot)$, etc. are defined as appropriate functions of X . For example, we let $r(t+1) = g_t(n_t)$ with a strictly positive and increasing function g_t for each t .

Let $\mathcal{S} = \{(t, n_t) : n_t = 0, 1, \dots, t; t = 0, 1, \dots, T\}$, where n_t denotes the number of up moves up to t in the binomial tree in Figure 7.2. The precise meaning of n_t is not crucial. It can be thought of as coding a number of exogenous factors that affects $r(\cdot)$, $B(\cdot, \cdot)$, etc. in a fixed income market. Assume that X is a Markov chain under an EMM Q , with transition probabilities

$$\begin{aligned} q(t, n_t) &= Q\{X(t+1) = (t+1, n_t+1) \mid X(t) = (t, n_t)\} \\ &= 1 - Q\{X(t+1) = (t+1, n_t) \mid X(t) = (t, n_t)\}, \end{aligned}$$

i.e. $n_{t+1} = n_t + 1$ or n_t are the only allowable transitions. Since $r(\cdot)$ is predictable, the value of spot rate $r(t+1)$ applied to the period $(t, t+1]$ is known at time t , denoted by $r(t+1, n_t)$.

Note that this formulation forces a recombining tree as shown in Figure 7.2, but this binomial tree is inhomogeneous in the sense that the transition probabilities q take different values at different nodes.

A full specification of the model involves $T(T+1)$ parameters: one value of q and one value of r at each node, with total number of $T(T+1)/2$ nodes (not counting the terminal nodes at T). The following special case illustrates this.

Example 7.3 Assume $q(t, n_t) \equiv \frac{1}{2} \forall (t, n_t) \in \mathcal{S}$. Moreover, the initial term structure $\{B(0, \tau), \tau = 1, \dots, T\}$ adds T constraints. There are still $T(T-1)/2$ parameters to be specified [among $T(T+1)/2$ values of r]. It is often useful to reparameterize the model by specifying the spot rate *volatilities*

$$\sigma(t, n_t) = \sqrt{q(t, n_t) [1 - q(t, n_t)]} \log \frac{r(t+2, n_t+1)}{r(t+2, n_t)} = \frac{1}{2} \log \frac{r(t+2, n_t+1)}{r(t+2, n_t)} \quad (7.12)$$

for $t = 0, 1, \dots, T-2$. [Is $\sigma(t, n_t) > 0 \forall (t, n_t)$? Why?]

If we let $B(t, n_t; \tau)$ denote the value of $B(t, \tau)$ at the state (t, n_t) , then (7.5) implies that

$$B(t, n_t; \tau) = \frac{q(t, n_t) B(t+1, n_t+1; \tau) + [1 - q(t, n_t)] B(t+1, n_t; \tau)}{1 + r(t+1, n_t)}. \quad (7.13)$$

7.4 Stochastic difference equations

Let the spot rate r be governed by the SDE

$$\Delta r(t+1) = \mu(t, r(t)) + \sigma(t, r(t)) \epsilon_t, \quad (7.14)$$

where μ is a real-valued function and σ a positive function; $\Delta r(t+1) = r(t+1) - r(t)$, and ϵ_t , $t = 0, 1, \dots, T-1$ are iid random variables with $Q(\epsilon_t = 1) = Q(\epsilon_t = -1) = \frac{1}{2}$. Obviously, under Q the increment $\Delta r(t+1)$ has the conditional mean $\mu(t, r(t))$ and the conditional variance $\sigma^2(t, r(t))$ given $r(t)$.

To see the connections between SDE and MC, first consider a special case of MC in which $q(t, n_t) \equiv \frac{1}{2} \forall (t, n_t) \in \mathcal{S}$. Then the increments $n_t - n_{t-1}$ are iid Bernoulli random variables under Q thus we can set

$$\epsilon_t = 2(n_t - n_{t-1}) - 1. \quad (7.15)$$

The SDE (7.14) is expressed as

$$r(t+1, n_{t-1} + 1) - r(t, n_{t-1}) = \mu(t, r(t, n_{t-1})) + \sigma(t, r(t, n_{t-1}))$$

and

$$r(t+1, n_{t-1}) - r(t, n_{t-1}) = \mu(t, r(t, n_{t-1})) - \sigma(t, r(t, n_{t-1})),$$

in which case μ and σ should satisfy

$$\mu(t, r(t, n_{t-1})) = \frac{1}{2} [r(t+1, n_{t-1} + 1) + r(t+1, n_{t-1}) - 2r(t, n_{t-1})]$$

and

$$\sigma(t, r(t, n_{t-1})) = \frac{1}{2} [r(t+1, n_{t-1} + 1) - r(t+1, n_{t-1})],$$

so that the spot rate r is governed by (7.14).

On the other hand, if r satisfies the SDE (7.14), then clearly r is a Markov chain. However, there are a number of more interesting and subtle questions to be answered:

Q1 Under what conditions on μ and σ , an underlying MC X can be represented by a recombining binomial tree as in Figure 7.2 such that r is defined by $r(t+1) = g_t(n_t)$?

Q2 What functions μ and σ will assure a positive spot rate r defined via (7.14)?

Q3 What functions μ and σ will guarantee that the spot rate r defined via (7.14) satisfies the *mean reversion* property, i.e. r tends to decrease when it is above a threshold and increase when it is below the threshold?

We will discuss these issues later in the studies of specific models.