Homework 1

Note: due Tue. 2/26 to get credit; will discuss these problems in class on Thur. 2/28.

(1) A combined spot/futures market contains a bank account \( B \), a stock \( S \) and a futures contract \( FU \) related to \( S \). Suppose the following data are collected:

\[
\begin{align*}
S(0) &= 1, \quad S(1) = 1.02, \quad S(2) = 0.99, \quad S(3) = 1.01, \quad \text{(unit price in dollar)}; \\
FU(0, 3) &= 1.04, \quad FU(1, 3) = 1.01, \quad FU(2, 3) = 1.02, \quad \text{(unit price in dollar)}; \\
B(0) &= 1, \quad r(1) = 0.001, \quad r(2) = r(3) = 0.002.
\end{align*}
\]

(1a) How much does it cost to take a long position of \( FU \) (with 100 shares) at \( t = 1 \)?

(1b) For a short position of \( FU \) (with 100 shares) taken at \( t = 2 \), what is the payoff at \( t = 3 \)?

(1c) Frank starts with $1,000 in \( B \). Suppose at \( t = 1 \), he shorts 100 shares of \( S \) and takes a long position of \( FU \) with 50 shares and the maturity date \( T = 3 \). What is the gain \( G(t) \) in Frank’s portfolio at \( t = 2 \) before marking to market? What is the value \( V(t) \) of his portfolio at \( t = 3 \)?

(2) Construct an example of arbitrage-free but incomplete single period model.

(3) Consider the binomial tree model: \( T = 3, \quad u = 1.07, \quad d = 0.92, \quad r = 6\% \) and \( S(0) = 2 \) (see the example in Lecture 2). Suppose a constant dividend yield \( \lambda = 5\% \) of the stock price is issued at the ex-dividend date \( t = 2 \).

(3a) For the chooser option (see Example 4.3) with the decision time \( T_0 = 2 \), the expiry \( T = 3 \), and the exercise price \( c = 2.05 \), construct a replicating portfolio and represent it in a binomial tree similar to Figure 2.3.

(3b) Calculate the value process of the American chooser option with the decision time \( T_0 = 1 \) and the exercise price \( c = 2.05 \).

(3c) Determine the optimal exercise strategy.

(4) Assume the stock price process starts with \( S(0) = 1 \) and follows a binomial tree with \( u = 1.08, \quad d = 0.91 \). Moreover, the short rate process is the same as in Example 7.1.

(4a) Consider a forward contract and a futures contract on the stock, both start at \( t = 0 \) and end at \( T = 3 \). Do they have the same value at \( t = 0 \)? Do they have the same price? Why?

(4b) For a European call option and a European put option on the futures \( FU(1, 3) \), specify the
exercise price $c$ such that these two European options have the same value at $t = 0$.

(5) Do Exercise 9.2.

(6) Consider several interest rate derivatives based on the model and data given in Example 7.1.

(6a) For $t = 0$, $\tau = 1$ and $\tau' = 2$, find the value of $\kappa$ such that $V_p(t) = V_r(t)$, i.e. the payer and receiver swaptions have the same value at $t = 0$ (assuming the underlying swap is ordinary and settled in advance).

(6b) For $\tau = 2$ and $\tau' = 3$, find the value of $\kappa$ such that the cap and floor have the same value at time $t = 0$ (both assumed to be ordinary and settled in arrears).

(6c) Consider a call caption and a put caption with the same expiration time $\tau = 2$ and the same exercise price $c$, based on the ordinary cap defined in (6b) [use your result in (6b) as the value of $\kappa$]. Find the value of $c$ such that the call and put captions have the same value at time $t = 0$. 
