(1) Let $X_1, \ldots, X_n$ be iid with $X_1 \sim N(\theta, 1)$. Consider the following two sets of hypotheses:

Set I: $H_0: \theta = 0$ vs $H_1: \theta \neq 0$;

Set II: $H_0: \theta = 0$ vs $H_1: |\theta| \geq 1$.

For which set it is easier to distinguish $H_1$ from $H_0$ based on $X_1, \ldots, X_n$? Justify your answer.

(2) Let $X$ have the density

$$f(x|\theta) = \begin{cases} \theta(x + \theta)^{-2}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

where $\theta > 0$ is an unknown parameter.
Construct an UMA-LCB of level \( 1 - \alpha \) for \( q(\theta) = \log \theta \) based on \( X \).

(3) Assume \( X_1, \ldots, X_n \) are iid from a common Poisson distribution with mean \( \lambda \). Let \( \theta = \lambda^2 \) and denote
\[
S_n = \sum_{i=1}^{n} X_i, \quad X^n = (X_1, \ldots, X_n).
\]
(a) Calculate the expectation \( E_{\theta} S_n^2 \).

(b) Find a minimal sufficient statistic for \( \theta \).

(c) Find a complete sufficient statistic for \( \theta \).

(d) Find the UMVUE \( T(X^n) \) for \( \theta \).
(e) Find the MLE $\hat{\theta}$ based on $X^n$.

(f) Calculate the Fisher information $I(\theta)$ associated with $X^n$.

(g) Is the Cramér-Rao lower bound for the variance of an unbiased estimator of $\theta$ attained? Justify your answer.

(4) Suppose $X \sim p_{\theta}$, $\theta \in \{0, 1, 2\}$, where $p_0(x) = \frac{1}{3}$, for $x = 0, 1, 2$; $p_1(1) = \frac{1}{3}$, $p_1(2) = \frac{2}{3}$; and $p_2(0) = \frac{1}{4}$, $p_2(2) = \frac{2}{3}$. 
(a) Find a UMP test of level \( \frac{1}{3} \) for \( H_0 : \theta = 0 \) vs \( H_1 : \theta = 1 \) or 2. (Hint: Consider two “simple vs simple” problems separately, then ...)

(b) Does there exist a UMP test of level \( \frac{2}{3} \)? Does a randomized test help? Why?

(5) Suppose \( X_1, ..., X_n \) are iid from a common Poisson distribution with mean \( \lambda \), and \( Y_1, ..., Y_n \) are iid from a common Poisson distribution with mean \( \mu \). Denote \( X^n = (X_1, ..., X_n) \) and \( Y^n = (Y_1, ..., Y_n) \). Assume \( X^n \) and \( Y^n \) are independent.

(a) Find a Wald test for \( H_0 : \lambda = \mu \) vs \( H_1 : \lambda \neq \mu \).
(b) Find a large sample confidence interval of level $1 - \alpha$ for $\theta = \lambda - \mu$.

(6) Let $\theta \in \{1, \ldots, 20\}$. Assume $P_{\theta}(X = x) = \frac{1}{\theta}$ for $x = 1, \ldots, \theta$. Show that for any function $g$, $g(X)$ is an admissible estimator of $g(\theta)$ under the squared error loss.

(7) For $n > 1$, let $X_1, \ldots, X_n$ be iid with common density

$$f(x|\theta) = \frac{e^x}{e^{\theta} - e^{-\theta}}, \quad x \in [-\theta, \theta],$$

where $\theta > 0$ is an unknown parameter. Write $X^n = (X_1, \ldots, X_n)$.

(a) Find a 1D minimal sufficient statistic for $\theta$ based on $X^n$. 
(b) Show that $T_n = \max\{|X_1|, \ldots, |X_n|\}$ is the MLE for $\theta$.

c) Show that $T_n$ is a consistent estimator with an exponential rate of convergence.

d) Assume $\theta$ follows a (prior) uniform distribution over $(0, 1)$. Conduct a Bayesian test based on a single observation $X_1 = 1/3$ for the hypotheses $H_0 : \theta \leq 2/3$ vs $H_1 : \theta > 2/3$. 