Lecture 2: Random Walk as a Probability Model

2.1 From coin tossing to stock fluctuation

The term “random walk” in the title of our textbook is perhaps intended as a cartoon for somewhat unpredictable fluctuations in stock markets. However, it also represents a class of probability toy models. In this lecture, we describe the simplest version of random walks and show how it is used for illustration of behaviors in stock markets.

Consider coin tossing — the simplest probability toy model. Suppose you have a coin with probabilities $P(H) = p$ and $P(T) = q$, where “H = head”, “T = tail”, and the positive constants $p$ and $q$ satisfy $p + q = 1$. In particular, the special case $p = q = 1/2$ corresponds to a fair coin. Otherwise the coin is said to be biased. Tossing a coin $n$ times repeatedly yields a $n$-string consisting of $n$ (H and/or T) letters. For any integer $k = 0, 1, \ldots, n$,

$$P(k \text{ H’s among } n \text{ tosses}) = \binom{n}{k} p^k q^{n-k}, \quad (\text{why?}) \quad (2.1)$$

which is referred to as the binomial probability formula.

Now assume an oversimplified stock as follows. In any trading day, the price of a stock has only two possible moves: up by an amount $\delta$ or down by the same amount $\delta$. The value of $\delta > 0$ is to be specified later. Here is a basic mathematical framework:

- $X_t$ = the increment of the stock price on day $t$, with $P(X_t = \delta) = p$ and $P(X_t = -\delta) = q$.
- $P_0$ = the initial stock price at $t = 0$.
- $S_n = X_1 + \cdots + X_n = \sum_{t=1}^{n} X_t$ = the cumulative increment of the stock price in $n$ days.
- $P_n = P_0 + S_n$ = the stock price on day $n$.

It follows immediately from the formula (2.1) that

$$P(\text{the stock price has } k \text{ up’s and } n - k \text{ down’s in } n \text{ days}) = \binom{n}{k} p^k q^{n-k}. \quad (2.2)$$

Various plots of this simplified stock price model will be shown in class along with discussion on related issues.
2.2 Q & A

(Q1) How are those plots generated?

(A1) Using “Matlab” with assigned values \( p = q = 1/2, \delta = 0.1 \) and \( n = 1,000 \) (trading days in about 4 years). However, 3 sets of plots will be shown with set 1 containing (daily) prices of all 1,000 days, set 2 containing (weekly) prices of 200 days, and set 3 containing (monthly) prices of 50 days. Matlab is not the only software that can do the job. In fact, Excel with Visual Basic incorporated in it can also perform Monte Carlo simulation and plot the results.

(Q2) Why the plots based on several runs of the same program look different?

(A2) The method that generates the plots is called Monte Carlo simulation. Every plot is an outcome of a random experiment, which occurs with a certain probability driven by the mechanism and the assigned values of \( p \) and \( q \). Each time you run the program, a new random number (seed) is generated. Different plots are due to different random numbers. The whole point is to use different random numbers to mimic the unpredictable stock fluctuations.

(Q3) What roles the parameters \( p, q \) and \( \delta \) play?

(A3) Here we just mention a few: (i) If \( p > 1/2 \) (or equivalently \( p > q \)), then the stock has a systematic upward trend, and vice versa; (ii) The greater the increment \( \delta \), the greater the stock volatility (bigger swing, greater risk, etc.) Trend and volatility are the most important characteristics of a stock. By changing numerical values of \( p, q \) and \( \delta \), we can generate stock plots with different trend and volatility behaviors.

(Q4) Do the simple random walk plots really look like real stock prices?

(A4) This is an important but difficult question, because the answer relies on how do we tell whether two plots (one for a real stock and the other simulated by a probability toy model) are “similar”. Some statistical methods provide partial answers. In general, more sophisticated probability models (beyond the scope of this course) are capable of producing plots that resemble many features of real stocks. Still, even the simple toy model given here helps us gain some basic understanding of stock markets.

(Q5) Are stock markets predictable?

(A5) The random walk theory advocated by our textbook says no. More specifically, suppose the toy model given here (with \( p = 1/2 \)) does mimic the typical behavior of a stock (call it stock A). Then we can only say that stock A will go up or down tomorrow with equal change 50%. Such (noninformative) prediction will remain the same even if we know stock A has been up for the past 10 trading days. In other words, any given information about the prices (or returns) of stock A up to now will not make stock A more or less likely to go up tomorrow. This is referred to as the statistical independence among stock A increments in different days.
Some financial economists challenged the random walk theory. See the book “A Non-Random
Walk Down Wall Street” by Andrew Lo and A. Craig MacKinlay (1999). But the debate goes on.