

# Lecture 1: Interest Rates

## 1.1 Time value of money

“\$100 today will be worth more than \$100 next year.” It simply means that you can let the value of \$100 grow by investing it. Although our FYS course focuses on stocks, it is important to have a basic understanding of other types of investment which can be compared to stocks. Suppose stock A and stock B have average annual returns 4% and 10% respectively, while a CD (certificate of deposit) account in your local bank offers 5% annual rate. (You are obligated to keep the account for 5 years, say.) Which one would you choose: A, or B, or CD? It need not be easy to decide due to many factors involved. Still, here is a simplified answer. Using statistical terms, both mean and variance need to be taken into consideration. On the one hand, the CD is better than stock A and worse than stock B in terms of mean profit. On the other hand, greater attention should be paid to variance (usually referred to as volatility in financial jargon). Here the CD is regarded as a bench mark for comparison since it is risk-free, i.e. \$100 deposited in the CD will for sure increase to \$105 after one year. The actual profitability margin in “stock B vs CD” could be much greater than the mean difference  $10\% - 5\% = 5\%$ , or it is also possible to lose a lot of money in stock B. Both outcomes have positive probabilities to occur. It is this uncertainty that makes investment decisions more challenging. Later on, we will discuss in more detail the issue of “return vs risk” — an inevitable and inseparable twin in investment. For one thing, if we want to compare the payoffs of stock B and the CD seriously, we at least need to know the chances for stock B price to go up and down (by a certain amount) in a given time period.

Albert Einstein once described compound interest as the “greatest mathematical discovery of all time.” (see our textbook, 2003 edition, p107.) Let  $r_a$  and  $r_m$  denote the annual and monthly interest rates. It is a useful exercise to convert  $r_a$  to an “equivalent”  $r_m$  in the sense that any initial value  $P_0$  will end up with the same amount, following two different compounding rules (annually and monthly), i.e.

$$P_0 (1 + r_a) = P_0 (1 + r_m)^{12}. \tag{1.1}$$

Cancelling  $P_0$  on both sides leads to

$$1 + r_a = (1 + r_m)^{12},$$

hence

$$r_m = \exp \left[ \frac{\log(1 + r_a)}{12} \right] - 1. \tag{1.2}$$

An approximation rule

$$r_m = r_a/12 \tag{1.3}$$

is often used. For  $r_a = 6\%$ , (1.2) and (1.3) will yield  $r_m = 0.48676\%$  and  $r_m = 0.5\%$  respectively. Another way to interpret (1.2) is that using the same interest rate in a given time period, a more frequent compounding rule will yield a greater return.

## 1.2 An example: mortgage loan

Frank bought a house in 1995 with market value \$130,000. He put \$30,000 down payment from his saving at the closing and borrowed \$100,000 in a 15-year loan with fixed annual (interest) rate 6%. To figure out his family budget, he calculated the required monthly payment, denoted by  $x$ , as follows:

Since 15 years = 180 months, Frank set the equation

$$x \left[ 1 + (1 + r_m) + (1 + r_m)^2 + \cdots + (1 + r_m)^{179} \right] = 100,000 (1 + r_m)^{180}. \tag{1.4}$$

Using  $r_m = 0.0048676$  calculated via (1.2) and the geometric sum formula (Can you recall it?), Frank had  $[\cdots] = \frac{(1+r_m)^{180}-1}{r_m} = 286.91335$ . Therefore,

$$x = \frac{100,000 (1 + 0.0048676)^{180}}{286.91335} = 835.2972823.$$

With Frank's family income, the monthly payment \$835.30 is manageable.

**Note:** To justify (1.4), notice that for each month  $n = 1, 2, \dots, 180$ ,

$$P_{n-1}(1 + r_m) - x = P_n,$$

where  $P_n$  represents the balance at the end of month  $n$ . Rewrite this as

$$x = P_{n-1}(1 + r_m) - P_n$$

then multiplying both sides by  $(1 + r_m)^{180-n}$  yields

$$x(1 + r_m)^{180-n} = P_{n-1}(1 + r_m)^{180-n+1} - P_n(1 + r_m)^{180-n}. \tag{1.5}$$

Adding all 180 equations (1.5) with  $n = 1, 2, \dots, 180$  together, and noting that  $P_0 = 100,000.00$  and  $P_{180} = 0.00$ , we will obtain (1.4). (Can you see all intermediate terms on the right hand sides cancel out?)

month	beginning balance	monthly payment	monthly interest	principal repayment	ending balance
1	100,000.00	835.30	486.76	348.54	99,651.46
2	99,651.46	835.30	485.06	350.24	99,301.22
...	...	...	...	...	...
96		835.30			
...	...	...	...	...	...
180		835.30			0.00

Table 1: Amortization schedule for a level-payment fixed-rate mortgage

**Homework 1:** (due Thur. 8/30)

- (a) How much interest Frank has to pay in addition to the principal \$100,000 he borrowed in the mortgage loan?
- (b) Suppose having followed this mortgage loan for 8 years, Frank did refinancing in 2003 by converting the original mortgage loan to a 5-year new loan with fixed annual rate 4%. What is Frank's monthly payment in the new loan?

Frank's refinancing might be motivated by a number of factors, such as an attractive low interest rate, the need for reducing the monthly payment, etc.