Lecture 4: Return vs Risk: Mean-Variance Analysis

4.1 Basics

We will discuss an important trade-off between return (or reward) — as measured by *expected return* — and risk — as measured by *variance of the return.* For simplicity, we only consider a single period model from time 0 to time 1. Suppose there are n assets in a portfolio. For each $i = 1, \dots, n$, let $X_i(0)$ and $X_i(1)$ be the time 0 and time 1 values of asset i respectively, and define the return of asset i in the time period to be

$$R_i = \frac{X_i(1) - X_i(0)}{X_i(0)}. \tag{4.1}$$

The portfolio can be represented by $(c_1, \dots, c_n)$, where c_i is the "proportion" of the initial capital held in asset i, $i = 1, \dots, n$. Note that some c_i could be negative, representing "shorting asset i". Still, we impose the constraint

$$\sum_{i=1}^{n} c_i = 1. \tag{4.2}$$

We can define the overall return R for the portfolio in the same manner, and have the relationship

$$R = \sum_{i=1}^{n} c_i R_i. \tag{4.3}$$

Imagine at time 0 an investor is trying to decide what portfolio will yield a high return R at time 1 while reducing the risk. Since R involves the future, it is regarded as a random variable. Some basic statistics for random variables will be introduced in Appendix, such as probability distribution, mean, variance, covariance and correlation. Those definitions and useful properties will be applied to several examples in Section 4.2.

4.2 Examples

See the notation and definitions introduced in Appendix. For asset i, let $\mu_i = ER_i$ and $\sigma_i^2 = Var(R_i)$ be the expected return (mean) and risk (variance). For assets i and j, let $\sigma_{ij} = Cov(R_i,\ R_j)$ denote the covariance between returns R_i and R_j. Note that $\sigma_{ii} = \sigma_i^2$.

Example 1 Assume that all returns are uncorrelated ($\sigma_{ij} = 0$ for all i and j) and have the same risk ($\sigma_i^2 = \sigma^2$ for all i). If we buy an equal number of shares for all n stocks in the portfolio ($c_i = 1/n$ for all i), then $Var(R) = \sigma^2/n$, i.e. the overall risk for the portfolio is only $1/n$ of the individual asset risk. The more uncorrelated different stocks included in the portfolio, the lower overall risk for the portfolio. This example illustrates that "diversification reduces risk".

Example 2 Consider a portfolio consisting of two assets. It follows from the property (P2) in Appendix that

$$Var(R) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + 2c_1c_2\,\sigma_{12},$$

which measures the portfolio risk. For simplicity we assume no borrowing, i.e. both c_1 and c_2 are positive. Given μ_1, μ_2, σ_1, σ_2, c_1 and c_2, consider $Var(R)$ in the following three cases:

Case 1: $\sigma_{12} < 0$. Placing negatively correlated assets in the same portfolio will reduce the portfolio risk — the idea of hedging.

Case 2: $\sigma_{12} = 0$. In a portfolio containing uncorrelated assets, the portfolio risk is a weighted sum of the individual asset risks. A suggestion: if it is not obvious how to find negatively correlated stocks when constructing your portfolio, you should at least include companies from those industries/sectors which only have loose connections.

Case 3: $\sigma_{12} > 0$. The portfolio risk will be greater than the weighted sum of the risks associated with those individual stocks if the portfolio consists of stocks that belong to the same industries/sectors, i.e. they tend to go up or down together.

Example 3 Consider a portfolio consisting of two perfectly correlated assets, i.e. $corr(R_1,\ R_2) = 1$ (or equivalently, $\sigma_{12} = \sigma_1\sigma_2$). Hence

$$Var(R) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + 2c_1c_2\,\sigma_{12} = (c_1\sigma_1 + c_2\sigma_2)^2.$$

Suppose $\mu_1 = 0.1$, $\sigma_1 = 0.08$, $\mu_2 = 0.14$, and $\sigma_2 = 0.18$. We can set $c_1 = 1.8$ and $c_2 = -0.8$, which means that for every \$100 invested, we long asset 1 with \$180 and short asset 2 with \$80. The expected return for the portfolio will be

$$ER = 1.8 \cdot 0.1 + (-0.8) \cdot 0.14 = 0.068.$$

But more importantly,

$$Var(R) = 0,$$

which implies that the portfolio is risk-free. In other words, to avoid an arbitrage opportunity, any available risk-free investment (such as a bank account) has to have a return rate equal to 0.068.

There is a trade-off between raising the expected return ER and reducing the risk $Var(R)$ for a portfolio. In general, you cannot achieve both at the same time. Here is an important portfolio optimization problem: Suppose we know μ_i, σ_i and σ_{ij} for all $i, j = 1, \ldots, n$. Given the mean return $\mu = ER$, construct a portfolio $(c_1, \ldots, c_n)$ such that its risk

$$\sigma^2 = Var(R) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \, \sigma_{ij}$$

is minimized. For $n \geq 3$, the problem can be solved by using calculus. Here we only consider the special case with two assets.

Eample 4 . Suppose $\mu_1 \neq \mu_2$. The two constraints $c_1\mu_1 + c_2\mu_2 = \mu$ and $c_1 + c_2 = 1$ yield $c_1 = \frac{\mu-\mu_2}{\mu_1-\mu_2}$ and $c_2 = \frac{\mu_1-\mu}{\mu_1-\mu_2}$. For example, with $\mu_1 = 0.15$, $\sigma_1 = 0.1$, $\mu_2 = 0.1$, $\sigma_2 = 0.1$ and $\sigma_{12} = 0$, we have $c_1 = 20\mu - 2$, $c_2 = 3 - 20\mu$ and

$$\sigma^2 = 4(\mu - 0.1)^2 + (0.3 - 2\mu)^2,$$

which demonstrates the relationship between μ and σ^2. The minimum risk $\sigma^2 = 0.005$ is attained when $\mu = 0.125$. In this case $c_1 = c_2 = 1/2$.

In practice, the parameters μ_i, σ_i, σ_{ij}, $i, j = 1, \ldots, n$ need not be given. They should be estimated based on historical data (see Homework 6).

4.3 Appendix

Assume X and Y are two random variables, each having a finite number of possible values: $\{x_1, \ldots, x_m\}$ for X and $\{y_1, \ldots, y_n\}$ for Y.

For a given function g of one variable, define the expectation

$$E[g(X)] = \sum_{i=1}^{m} g(x_i)P(X = x_i). \tag{4.4}$$

Using various functions g enables us to evaluate important features regarding X. Here are two of them.

$$\mu = EX \quad = \quad \sum_{i=1}^{m} x_i P(X = x_i) \quad \text{(mean of } X\text{)}; \tag{4.5}$$

$$\sigma^2 = Var(X) \quad = \quad E\left[(X - \mu)^2\right] = E(X^2) - \mu^2 \quad \text{(variance of } X\text{)}. \tag{4.6}$$

$\sigma = SD(X) = \sqrt{Var(X)}$ is called the standard deviation of X.

Similarly, for a given function G of two variables, define the expectation

$$E[G(X,Y)] = \sum_{i=1}^{m}\sum_{j=1}^{n} G(x_i, y_j)\ P(X = x_i, Y = y_j). \tag{4.7}$$

In particular, the covariance between X and Y is defined by

$$Cov(X,Y) = E[(X - EX)(Y - EY)] = E(XY) - EX\ EY. \tag{4.8}$$

Moreover, the correlation between X and Y is defined by

$$corr(X,Y) = \frac{Cov(X,Y)}{SD(X)\ SD(Y)}. \tag{4.9}$$

EX and $Var(X)$ are summaries of the probability distribution of X, measuring the central tendency and dispersion respectively. $Cov(X,Y)$ or $corr(X,Y)$ is a summary of the joint probability distribution of X and Y, measuring certain statistical dependence between X and Y.

X and Y are said to be (statistically) independent if $P(X = x_i, Y = y_j) = P(X = x_i)\ P(Y = y_j)$ for any possible pair of (x_i, y_j), $i = 1, ..., m$, $j = 1, ..., n$. X and Y are said to be uncorrelated if $Cov(X,Y) = corr(X,Y) = 0$. Note that if X and Y are independent, then they are uncorrelated. But the converse need not be true. Hence "correlation zero" is a weaker version of independence.

Here is a list of useful properties, where X, Y, Z are random variables and c, c_1, c_2 are constants.

(P1) $E(c) = c, \quad E(cX) = c\ EX$ and $E(X+Y) = EX + EY$.

(P2) $Var(c) = 0, \quad Var(cX) = c^2\ Var(X)$ and $Var(X+Y) = Var(X) + Var(Y) + 2\ Cov(X,Y)$.

(P3) $Cov(X,X) = Var(X)$ and $Cov(c_1X,\ c_2Y) = c_1c_2\ Cov(X,Y)$.

(P4) $Cov(X+Y,\ Z) = Cov(Z,\ X+Y) = Cov(X,Z) + Cov(Y,Z)$.

(P5) $-1 \le corr(X,Y) \le 1$.

(P6) If X and Y are independent, then $Cov(X,Y) = corr(X,Y) = 0$.

Homework 5: (due Tuesday 10/23)

(1) Assume that the particular portfolio given in Example 3 is available to you in two consecutive time periods, along with a local bank account with interest rate 5% in the 1st period and 7% in the 2nd period. What would be your investment strategy in these two periods?

(2) Consider a portfolio with three assets in it. Assume $\mu_1 = 0.1$, $\mu_2 = 0.2$, $\mu_3 = 0.3$ and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 0.01$. Suppose you have a grand total \$1,000 to allocate in the portfolio subject to the constraint that you have to put \$200 in asset 1. Explain how should you allocate the remaining wealth (\$800) in assets 2 and 3 in the following two cases respectively, in order to obtain an expected return 0.22 for the portfolio:

Case 1: $\sigma_{23} = 0$;

Case 2: $\sigma_{23} = -0.5$.

Comment on your findings.