

Lecture 12 Some Extensions of Neyman-Pearson Theory

12.1 One-sided hypotheses and monotone likelihood ratios

Definition 1 $\mathcal{F} = \{f(\cdot|\theta), \theta \in \Theta\}$ is said to have a monotone likelihood ratio (MLR) for a sufficient statistic U if for any $\theta_0 < \theta_1$, the ratio $\frac{f(x|\theta_1)}{f(x|\theta_0)} = \frac{g(U(x)|\theta_1)}{g(U(x)|\theta_0)}$ (see Theorem 1 in Lecture 4) is nondecreasing in U on the set $\{x \in \mathcal{X} : g(U(x)|\theta_0) > 0 \text{ or } g(U(x)|\theta_1) > 0\}$.

Theorem 1 (Karlin-Rubin Theorem, see Casella and Berger, p391, Theorem 8.3.17 for the proof.)

Consider testing $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$. Suppose there exists a MLR for U . Then there exists an UMPT $d \in \mathcal{T}_\alpha$ such that

- (i) the power function $\beta(\theta, d)$ is nondecreasing in θ and $\beta(\theta_0, d) = \alpha$;
- (ii) the test function

$$d(x) = \begin{cases} 1, & \text{if } U(x) > c, \\ 0, & \text{if } U(x) < c, \\ \gamma, & \text{if } U(x) = c \text{ (possible randomization),} \end{cases}$$

where $c \in \mathbb{R}$ and $\gamma \in [0, 1]$ are determined by the condition $P_{\theta_0}(U > c) + \gamma P_{\theta_0}(U = c) = \alpha$.

Conversely, any level- α UMPT will satisfy (i), and essentially be of the form in (ii).

Example 12.1 Let X_1, \dots, X_n be iid random variables with $X_1 \sim f(x|\theta) = \exp[\theta T(x) - \psi(\theta)]$. Then there exists a MLR for $U = \sum_{i=1}^n T(X_i)$.

Example 12.2 Let X_1, \dots, X_n be iid $\mathcal{U}(0, \theta)$ random variables. Then there exists a MLR for $X_{(n)}$ and no need for randomized tests.

Example 12.3 (sign test) Let X_1, \dots, X_n be iid random variables with common density $f(x-\theta)$, where f is a continuous even function. Both θ and f are unknown. Consider testing $H_0 : \theta \leq 0$ vs $H_1 : \theta > 0$. Note in this case that f can be treated as an infinite dimensional nuisance parameter due to the following transformation. Set $\xi = P_{\theta, f}(X_1 \geq 0)$. Then “ H_0 vs H_1 ” is equivalent to testing $H'_0 : \xi \leq 1/2$ vs $H'_1 : \xi > 1/2$. Let $U_i = I_{\{X_i \geq 0\}}$, $i = 1, \dots, n$. Then U_1, \dots, U_n are iid Bernoulli random variables with parameter ξ , and $T \triangleq \sum_{i=1}^n U_i$

follows a binomial distribution $Bin(n, \xi)$. Theorem 1 implies that the test function

$$d(X^n) = \begin{cases} 1, & \text{if } T > c, \\ 0, & \text{if } T < c, \\ \gamma, & \text{if } T = c \text{ (possible randomization),} \end{cases}$$

is an UMPT for H'_0 vs H'_1 , hence an UMPT for H_0 vs H_1 , where c is an integer that satisfies $P_{1/2}(T > c) \leq \alpha \leq P_{1/2}(T \geq c)$. Note that $T > c \iff X_{(n-c)} \geq 0$. The sign test is distribution-free, since $\beta(\theta, d)$ does not depend on f under H_0 .

12.2 Two-sided hypotheses and unbiased tests

Example 12.4 Let X_1, \dots, X_n be iid $N(\theta, 1)$ random variables. Consider testing $H_0 : \theta = 0$ vs $H_1 : \theta \neq 0$. Claim: No level- α UMPT exists. Here is why.

Consider two “simple vs simple” problems:

- (i) $H_0 : \theta = 0$ vs $H'_1 : \theta = \theta'$ and
- (ii) $H_0 : \theta = 0$ vs $H''_1 : \theta = \theta''$

with two arbitrary constants $\theta' > 0$ and $\theta'' < 0$. Let d' and d'' denote the UMPTs for (i) and (ii) respectively, with (“extended”) power functions $\beta(\theta, d') = 1 - \Phi\left(\frac{c-\theta}{\sqrt{n}}\right)$ and $\beta(\theta, d'') = \Phi\left(\frac{-c-\theta}{\sqrt{n}}\right)$. If d is an UMPT for H_0 vs H_1 , then d is also an UMPT for both (i) and (ii). (why? — the power function of d can be evaluated at any $\theta \in \mathbb{R}$.) Neyman-Pearson Lemma implies essentially that $d = d'$ and $d = d''$ — contradiction!

A remedy is to consider only “unbiased tests”.

Definition 2 Consider testing $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$.

- (i) d is called an unbiased test if $\sup_{\theta \in \Theta_0} \beta(\theta, d) \leq \inf_{\theta \in \Theta_1} \beta(\theta, d)$.
- (ii) For $\alpha \in (0, 1)$, let \mathcal{UT}_α be the set of all unbiased level- α tests. d is called an uniformly most powerful unbiased test (UMPUB) if

$$\beta(\theta, d) \geq \beta(\theta, d') \quad \forall \theta \in \Theta_1 \text{ and } \forall d' \in \mathcal{UT}_\alpha.$$

Proposition 1 Let X_1, \dots, X_n be iid random variables with $X_1 \sim f(x|\theta) = \exp[\theta T(x) - \psi(\theta)]$. For testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$, the following test d is an UMPUB:

$$d(X^n) = \begin{cases} 0, & \text{if } U \in (c_1, c_2), \\ 1, & \text{if } U \notin [c_1, c_2], \\ \gamma_i, & \text{if } U = c_i, \quad i = 1, 2 \text{ (possible randomization),} \end{cases}$$

where $U = \sum_{i=1}^n T(X_i)$, and $c_1, c_2, \gamma_1, \gamma_2$ are determined by

$$(i) E_{\theta_0} [d(X^n)] = \alpha;$$

$$(ii) E_{\theta_0} [d(X^n) U] = \alpha E_{\theta_0} U.$$

Here is a heuristic argument, not a rigorous proof: Consider a LR test. Note that for $\theta \neq \theta_0$, we reject H_0 if the LR $\frac{f(x^n|\theta)}{f(x^n|\theta_0)} = \exp\{(\theta - \theta_0) U(x^n) - n[\psi(\theta) - \psi(\theta_0)]\}$ is large, i.e.

$$\begin{cases} U < c_1, & \text{if } \theta < \theta_0, \\ U > c_2, & \text{if } \theta > \theta_0. \end{cases}$$

To see why (ii) holds, note that (i) implies

$$\begin{aligned} \frac{d\beta(\theta, d)}{d\theta} &= \frac{d}{d\theta} \int d(x^n) \exp[\theta U(x^n) - n\psi(\theta)] \nu(dx^n) \\ &= \int d(x^n) \left[U(x^n) - n \frac{d\psi(\theta)}{d\theta} \right] \exp[\theta U(x^n) - n\psi(\theta)] \nu(dx^n) \\ &= E_{\theta} \left\{ d(X^n) \left[U(X^n) - n \frac{d\psi(\theta)}{d\theta} \right] \right\}. \end{aligned}$$

Since d is unbiased, $\beta(\theta_0, d) \leq \beta(\theta, d) \forall \theta$. Hence $\left. \frac{d\beta(\theta, d)}{d\theta} \right|_{\theta=\theta_0} = 0$, and

$$E_{\theta_0} [d(X^n) U] = E_{\theta_0} d(X^n) \left[n \left. \frac{d\psi(\theta)}{d\theta} \right|_{\theta=\theta_0} \right] = \alpha E_{\theta_0} U.$$

Example 12.4 (revisited) Following Proposition 1, we reject H_0 if and only if $S_n \neq [c_1, c_2]$, with $P_0(S_n \neq [c_1, c_2]) = \alpha$ and $E_0 \left\{ S_n I_{\{S_n \neq [c_1, c_2]\}} \right\} = 0$. In particular, we have $c_2 = -c_1 \triangleq c = \sqrt{n} z_{\alpha/2}$. (why?)