Lecture 13 Confidence Intervals by Inverting Test Statistics

13.1 Basic elements in confidence intervals

Definition 1 *For* $\alpha \in (0,1)$, $[l(X),\ u(X)]$ *is called a level* $1-\alpha$ *(or* $100(1-\alpha)\%$ *) confidence interval (CI) for* $q(\theta)$ *if*

$$\inf_{\theta\in\Theta} P_\theta(l(X) \le q(\theta) \le u(X)) \ge 1-\alpha.$$

Note:

- Terms:

  coverage probability $P_\theta(l(X) \le q(\theta) \le u(X))$

  confidence level $1-\alpha$

  lower confidence bound (LCB) $l(X)$ (when $u(X) = \infty$)

  upper confidence bound (UCB) $u(X)$ (when $l(X) = -\infty$)

- In general, a subset $C(X) \subset \widetilde{\Theta} \triangleq q(\Theta)$ is referred to as a level $1-\alpha$ *confidence set* for $q(\theta)$ if

  $$\inf_{\theta\in\Theta} P_\theta(q(\theta) \in C(X)) \ge 1-\alpha.$$

- There is a trade-off for a CI between its coverage probability (reliability) and its length (accuracy). Based on a finite sample X, we usually cannot improve one aspect without sacrificing the other.

- If $l(X) < u(X)$ are level $1-\alpha$ LCB and UCB for $q(\theta)$ respectively, then $[l(X),\ u(X)]$ is a level $1-2\alpha$ CI for $q(\theta)$. (why?)

- Interpretation of CI: Given data $x \in \mathcal{X}$, it is not literally correct to say "$[l(x),\ u(x)]$ covers $q(\theta)$ with probability $1-\alpha$". A correct understanding should be this: If we apply the same procedure $[l(\cdot),\ u(\cdot)]$ on repeated samples $x^{(1)},...,x^{(m)}$, then the relative frequency for $[l(x^{(i)}),\ u(x^{(i)})]$, $i = 1,...,m$ to cover $q(\theta)$ will be approximately $1-\alpha$ for large m.

13.2 Duality between hypothesis testing and CIs

To illustrate the idea, we only consider non-randomized tests and let $q(\theta) = \theta$.

For every $\theta_0 \in \Theta$, let $A(\theta_0) = \{x \in \mathcal{X} : \ d_{\theta_0}(x) = 0\}$ be the acceptance region of a test d_{θ_0} for $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. Define $C(x) = \{\theta \in \Theta : \ d_\theta(x) = 0\}$ for every $x \in \mathcal{X}$. It follows from "$\theta \in C(x) \iff x \in A(\theta)$" that $C(X)$ is a level $1-\alpha$ confidence set for θ if and only if d_θ is a level-α test for H_0 vs H_1.

Note:

- The above duality only offers a guideline for obtaining a confidence set $C(X)$ with no guarantee that $C(X)$ is actually a CI $[l(X),\ u(X)]$, which should be checked in each particular problem.
- $l(X)$ is a LCB for $\theta \iff \forall\ \theta_0 \in \Theta, d_{\theta_0}(X) = 1$ when $l(X) > \theta_0$ for $H_0 : \theta = \theta_0$ vs $H_1 : \theta > \theta_0$;
- $u(X)$ is an UCB for $\theta \iff \forall\ \theta_0 \in \Theta, d_{\theta_0}(X) = 1$ when $u(X) < \theta_0$ for $H_0 : \theta = \theta_0$ vs $H_1 : \theta < \theta_0$.

Example 13.1 Let $X_1, ..., X_n$ be iid samples from $\mathcal{U}(0, \theta)$. For $H_0 : \theta = \theta_0$ vs $H_1 : \theta > \theta_0$, we have an UMPT

$$d(x^n) = \begin{cases} 1, & \text{if } x_{(n)} > c \\ 0, & \text{if } x_{(n)} \leq c, \end{cases}$$

where $c = \theta_0(1-\alpha)^{1/n}$. Therefore, $l(X^n) = \frac{X_{(n)}}{(1-\alpha)^{1/n}}$ is a level $1-\alpha$ LCB for θ, i.e.

$$P_\theta(X_{(n)} \leq \theta\ (1-\alpha)^{1/n}) = P_\theta\left(\theta \geq \frac{X_{(n)}}{(1-\alpha)^{1/n}}\right) = 1-\alpha, \quad \forall\ \theta > 0.$$

Example 13.2 Let $X_1, ..., X_n$ be iid samples from an exponential distribution with common density $f(x|\theta) = \theta\ e^{-\theta x}$, $x > 0$. To construct a level $1-\alpha$ CI for $\theta > 0$, we invert the acceptance region of a LR test for $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. The LR test statistic

$$\Lambda(x^n) = \frac{(\hat{\theta})^n\ e^{-\hat{\theta} s_n}}{\theta_0^n\ e^{-\theta_0 s_n}} = \left[(\theta_0 \bar{x})\ e^{1-\theta_0 \bar{x}}\right]^{-n},$$

where $\hat{\theta} = (\bar{x})^{-1}$. Let $T = \theta_0 \bar{X}$. Then the acceptance is specified by

$$\begin{aligned} & \Lambda(X^n) \leq \text{ constant} \\ \iff\ & T\ e^{1-T} \geq \text{constant} \\ \iff\ & T\ e^{-T} \geq \text{constant} \\ \iff\ & g(T) \stackrel{\triangle}{=} \log T - T \geq \text{constant}. \end{aligned}$$

Note that $g(0) = g(\infty) = -\infty$, and $\frac{dg}{dT} = T^{-1} - 1$, which is "$> 0$" if $T < 1$, "$= 0$" if $T = 1$ and "< 0" if $T > 1$. It implies that $g(1) = -1 = \max_{T>0} g(T)$. Therefore, $[b_1,\ b_2]$ is a CI for θ where b_1 and b_2 satisfy $g(b_1) = g(b_2)$ and $P_{\theta_0}(b_1 \le T \le b_2) = 1 - \alpha$. Note that $T = \frac{\theta_0 S_n}{n}$ where S_n follows a gamma distribution $\mathcal{G}(n, \theta_0)$ with density $f(x|n, \theta_0) = \frac{\theta_0^n\ x^{n-1}\ e^{-\theta_0 x}}{\Gamma(n)}$. A valid CI relies on that b_1 and b_2 should not depend on θ_0, which is indeed the case. (why?) This fact has a broader implication. See Lecture 14 for further elaboration.