Lecture 15  Certain Optimality of Confidence Intervals

One way to evaluate a confidence set is to measure its probability of false coverage, i.e. the probability of covering false parameter values, which is similar to a type II error probability in hypothesis testing.

Definition 1 Let $\Theta' \subset \Theta$ and $\theta \in \Theta \backslash \Theta'$. A level $1 - \alpha$ confidence set $C(X)$ for $\theta$ is said to be $\Theta'$-uniformly most accurate (UMA) if for any level $1 - \alpha$ confidence set $C'(X)$, we have

$$P_\theta(\theta' \in C(X)) \leq P_\theta(\theta' \in C'(X)), \quad \forall \theta' \in \Theta'.$$

Here are some special cases for 1D $\Theta$:

(i) UMA-CI: $\Theta' = \Theta \backslash \{\theta\}$, $C(X) = [l(X), u(X)]$.
(ii) UMA-LCB: $\Theta' = (-\infty, \theta)$, $C(X) = [l(X), \infty)$.
(iii) UMA-UCB: $\Theta' = (\theta, \infty)$, $C(X) = (-\infty, u(X)]$.

Definition 2 Let $\Theta' \subset \Theta$ and $\theta \in \Theta \backslash \Theta'$. A level $1 - \alpha$ confidence set $C(X)$ is said to be $\Theta'$-unbiased if

$$P_\theta(\theta' \in C(X)) \leq 1 - \alpha \quad \forall \theta' \in \Theta'.$$

Note: Uniformly most accurate unbiased CI (UMAU-CI), UMAU-LCB and UMAU-UCB can be defined in an obvious manner.

Theorem 1 Consider non-randomized tests for $H_0 : \theta = \theta_0$ vs $H_1 : \theta > \theta_0$. Suppose a level-$\alpha$ UMPT is given by

$$d_{\theta_0}^\ast(x) = \begin{cases} 
1, & \text{if } l^\ast(x) > \theta_0, \\
0, & \text{if } l^\ast(x) \leq \theta_0,
\end{cases}$$

then $l^\ast(X)$ is a level $1 - \alpha$ UMA-LCB for $\theta$.

Proof: Let $l(X)$ be a level $1 - \alpha$ LCB for $\theta$. Then

$$d_{\theta_0}(x) = \begin{cases} 
1, & \text{if } l(x) > \theta_0, \\
0, & \text{if } l(x) \leq \theta_0
\end{cases}$$

is a level-$\alpha$ test for $H_0$ vs $H_1$. Hence for $\theta > \theta_0$,

$$P_\theta(l^\ast(X) \leq \theta_0) = 1 - \beta(\theta, d_{\theta_0}^\ast) \leq 1 - \beta(\theta, d_{\theta_0}) = P_\theta(l(X) \leq \theta_0).$$

QED.
Note:

- A similar result can be established for UMA-UCB via the following relationship: $u(X)$ is a level $1 - \alpha$ UCB for $\theta$ if and only if $-u(X)$ is a level $1 - \alpha$ LCB for $-\theta$.
- Many results derived before for UMPT with exponential families, MLRs, etc. can be translated to their counterparts for UMA confidence bounds.

Example 15.1 (Example 13.1 revisited) Let $X_1, \ldots, X_n$ be iid samples from $U(0, \theta)$. For $H_0 : \theta = \theta_0$ vs $H_1 : \theta > \theta_0$, we have an UMPT

$$d(x^n) = \begin{cases} 
1, & \text{if } x_{(n)} > c \\
0, & \text{if } x_{(n)} \leq c,
\end{cases}$$

where $c = \theta_0(1 - \alpha)^{1/n}$. Therefore, the test $d$ can be rewritten as

$$d(x^n) = \begin{cases} 
1, & \text{if } l(x^n) > \theta_0 \\
0, & \text{if } l(x^n) \leq \theta_0
\end{cases}$$

with $l(X^n) = X_{(n)}^{(1 - \alpha)^{1/n}}$. By Theorem 1, $l(X^n)$ is a level $1 - \alpha$ UMA-LCB for $\theta$.

The following theorem is a counterpart of two-sided UMPT.

**Theorem 2** Consider non-randomized unbiased tests for $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. Suppose $A^*(\theta_0)$ is the acceptance region of a level-\(\alpha\) UMPUT $d_{\theta_0}^*$, and $C^*(X)$ is the level $1 - \alpha$ confidence set obtained by inverting $A^*(\theta_0)$. Then $C^*(X)$ is an UMAU-confidence set for $\theta$. In particular, if $C^*(X)$ is an interval, then it is an UMAU-CI for $\theta$.

**Proof:** Let $C(X)$ be an unbiased level $1 - \alpha$ confidence set for $\theta$. Then

$$d_{\theta_0}(x) = \begin{cases} 
1, & \text{if } \theta_0 \notin C(X) \\
0, & \text{if } \theta_0 \in C(X)
\end{cases}$$

is a level-\(\alpha\) unbiased test for $H_0$ vs $H_1$. Hence for every $\theta \neq \theta_0$,

$$P_\theta(\theta_0 \in C^*(X)) = P_\theta(A^*(\theta_0)) = 1 - P_\theta(A^*(\theta_0)) = 1 - \beta(\theta, d_{\theta_0}^*) \leq 1 - \beta(\theta, d_{\theta_0}) = P_\theta(A(\theta_0)) = P_\theta(\theta_0 \in C(X)).$$

**QED.**

Here is a different view on UMA CIs and confidence bounds.
Proposition 1 Fix $\alpha \in (0, 1)$.

(i) Let $l^*(X)$ and $u^*(X)$ be level $1 - \alpha$ UMA-LCB and UMA-UCB respectively. Then for every $\theta$ and any level $1 - \alpha$ LCB $l(X)$ and UCB $u(X)$,
\[
E_\theta[\theta - l^*(X)]^+ \leq E_\theta[\theta - l(X)]^+ \quad \text{and} \quad E_\theta[u^*(X) - \theta]^+ \leq E_\theta[u(X) - \theta]^+.
\]

(ii) Let $[l^*(X), u^*(X)]$ be a level $1 - \alpha$ UMAU-CI for $\theta$. Then for every $\theta$ and any level $1 - \alpha$ unbiased CI $[l(X), u(X)]$ for $\theta$,
\[
E_\theta[u^*(X) - l^*(X)]^+ \leq E_\theta[u(X) - l(X)]^+.
\]

Proof: The first inequality in (i) for LCB follows from
\[
E_\theta[\theta - l^*(X)]^+ = \int_0^\infty P_\theta(\theta - l^*(X) \geq y) \, dy
\]
\[
= \int_0^\infty P_\theta(l^*(X) \leq \theta - y) \, dy
\]
\[
\leq \int_0^\infty P_\theta(l(X) \leq \theta - y) \, dy
\]
\[
= E_\theta[\theta - l(X)]^+.
\]

The same argument will yield the second inequality in (i) for UCB.

For every $\theta \in \Theta$, (ii) follows from
\[
E_\theta[u^*(X) - l^*(X)]^+ = \int_X [u^*(x) - l^*(x)]^+ f(x|\theta) \, \nu_X(dx)
\]
\[
= \int_X \left[ \int_{\Theta \setminus \{\theta\}} I_{\{l^*(x) \leq \theta' \leq u^*(x)\}} \, d\theta' \right] f(x|\theta) \, \nu_X(dx)
\]
(by Fubini Theorem)
\[
= \int_{\Theta \setminus \{\theta\}} \left[ \int_X I_{\{l^*(x) \leq \theta' \leq u^*(x)\}} f(x|\theta) \, \nu_X(dx) \right] \, d\theta'
\]
\[
= \int_{\Theta \setminus \{\theta\}} P_\theta(l^*(X) \leq \theta' \leq u^*(X)) \, d\theta'
\]
\[
\leq \int_{\Theta \setminus \{\theta\}} P_\theta(l(X) \leq \theta' \leq u(X)) \, d\theta'
\]
\[
= E_\theta[u(X) - l(X)]^+.
\]

QED.