Final (Stor 635)
May 1, 2008

The final contains 9 problems concerning conditioning and martingales. Each problem is worth 3 points. Please write partial solutions or consider special cases. Keep in mind that the final is closed book and closed mouth. Good luck!

Problem 1. Prove the following basic properties of conditional expectations $E(\xi|\mathcal{G})$: (*i*) Existence of $E(\xi|\mathcal{G})$ follows as a simple application of Radon-Nikodym theorem. Provide an exact argument for this application. (*ii*) If $\mathcal{G}_2 \subset \mathcal{G}_1$, then $E(E(\xi|\mathcal{G}_1)|\mathcal{G}_2) = E(\xi|\mathcal{G}_2) = E(E(\xi|\mathcal{G}_2)|\mathcal{G}_1)$ a.s. (*iii*) If $\sigma(\xi)$ and $\mathcal{G}$ are independent, then $E(\xi|\mathcal{G}) = E\xi$ a.s.

Problem 2. (*i*) Let $\Omega = [0,1)$, $\mathcal{F} = \mathcal{B}[0,1)$, $\mathcal{G} = \sigma\{[0,1/3),[1/3,2/3),[2/3,1)\}$, $P =$ Lebesgue measure and $\xi(w) = w$, $w \in \Omega$. Compute $E(\xi|\mathcal{G})$ and find a regular conditional probability $P(A,w)$ on $\mathcal{F}$ given $\mathcal{G}$. (*ii*) Suppose a random vector (ξ,η) has a joint p.d.f.

$$f(x,y) = \begin{cases} x+y & \text{if } 0<x,y<1, \\ 0 & \text{otherwise.} \end{cases}$$

What are a regular conditional density $\widehat{f}_{\xi|\eta}(x,y)$ of ξ given $\eta = y$, and a regular conditional distribution $\widehat{Q}_{\xi|\eta}(B,y)$ of ξ given $\eta = y$? What about a regular conditional density $f_{\xi|\eta}(x,w)$ of ξ given η, and a regular conditional distribution $Q_{\xi|\eta}(B,w)$ of ξ given η? Compute also $E(\xi|\eta = y)$, $E(\xi|\eta)$ and $E(\xi^2|\eta = y)$.

Problem 3. Let $\xi \in L^2(\Omega,\mathcal{F},P)$ and $\mathcal{F} \supset \mathcal{G}$ be a sub-σ-field. Show that $E(\xi|\mathcal{G})$ is the unique r.v. $\eta \in L^2(\Omega,\mathcal{F},P)$ which minimizes $E(\xi-\eta)^2$ and that the minimum value is $E\xi^2 - E(E(\xi|\mathcal{G})^2)$.

Problem 4. Let ξ,η,ζ be r.v.'s with $E|\xi| < \infty$ and η independent of the pair (ξ,ζ). Show that

$$E(\xi|\eta,\zeta) = E(\xi|\zeta) \quad \text{a.s.}$$

Problem 5. Let $\{\xi_i^n, i,n \geq 1\}$ be i.i.d. nonnegative, integer-valued r.v.'s. Consider a branching process $\{Z_n\}_{n\geq 1}$ defined by $Z_0 = 1$ and

$$Z_{n+1} = \begin{cases} 0 & \text{if } Z_n = 0, \\ \xi_1^{n+1} + \ldots + \xi_{Z_n}^{n+1} & \text{if } Z_n > 0. \end{cases}$$

Let $\mathcal{F}_n = \sigma\{\xi_i^m, i \geq 1, m \leq n\}$ and suppose $\mu = E\xi_i^m \in (0,\infty)$. Show that $\{Z_n/\mu^n, \mathcal{F}_n\}$ is a martingale.

Problem 6. Doob's decomposition states that every submartingale $\{\xi_n,\mathcal{F}_n\}$ can be uniquely decomposed as

$$\xi_n = \eta_n + \zeta_n \quad \text{all } n, \text{ a.s.,}$$

where $\{\eta_n,\mathcal{F}_n\}$ is a martingale and the r.v.'s $\{\zeta_n\}$ are such that $\zeta_1 = 0$ a.s., $\zeta_n \leq \zeta_{n+1}$ a.s. all n, and ζ_{n+1} is $\mathcal{F}_n$-measurable for all n.

(*i*) The above properties, in fact, immediately determine $\{\eta_n\}$ and $\{\zeta_n\}$ in terms of the submartingale $\{\xi_n,\mathcal{F}_n\}$. Find these expressions of $\{\eta_n\}$ and $\{\zeta_n\}$ in terms of $\{\xi_n,\mathcal{F}_n\}$. (You may want to start with $n = 1,2$ and proceed from there.) (*ii*) Prove that the Doob's decomposition is unique.

Problem 7. Fill in the numbered boxes below in the proof of a.s. convergence of a submartingale, taken from the lecture notes.

Theorem 14.3.1 Let $\{\xi_n, \mathcal{F}_n\}$ be a submartingale. If

$$\lim_{n\to\infty} \boxed{1} < \infty$$

then there is an integrable r.v. ξ_∞ such that $\xi_n \to \xi_\infty$ a.s.

Proof: For every pair of real numbers $a < b$, let $U_{[a,b]}^{(n)}(w)$ be the number of $\boxed{2}$ of $[a, b]$ by $\{\xi_i(w) : 1 \le i \le n\}$. Then $\{U_{[a,b]}^{(n)}(w)\}$ is a $\boxed{3}$ sequence of random variables and thus has a limit

$$U_{[a,b]}(w) = \lim_{n\to\infty} U_{[a,b]}^{(n)}(w) \quad \text{a.s.}$$

By $\boxed{4}$ and $\boxed{5}$, we have

$$EU_{[a,b]} = \lim_{n\to\infty} EU_{[a,b]}^{(n)} \le \lim_{n\to\infty} \boxed{6} < \infty,$$

so that $U_{[a,b]}\boxed{7}$ a.s. It follows that if

$$E_{[a,b]} = \{w \in \Omega : \liminf_{n\to\infty} \xi_n(w) < a < b < \limsup_{n\to\infty} \xi_n(w)\}$$

then $P(E_{[a,b]}) = \boxed{8}$ for all $a < b$. Thus if

$$E = \bigcup_{\boxed{9}} E_{[a,b]} = \{w \in \Omega : \liminf_{n\to\infty} \xi_n(w) < \limsup_{n\to\infty} \xi_n(w)\}$$

then $P(E) = \boxed{10}$. It follows that $\liminf_{n\to\infty} \xi_n(w) = \limsup_{n\to\infty} \xi_n(w)$ a.s. and thus the limit $\lim \xi_n$ exists a.s. Denote this limit by ξ_∞. Then, by Fatou's lemma, $E|\xi_\infty| \le \liminf_n \boxed{11}$ and since $E|\xi_n| = \boxed{12} \le 2E\xi_n^+ - E\xi_1$, we obtain that $E|\xi_\infty| < \infty$. Thus ξ_∞ is integrable.

Problem 8. Let Ω be the set of all positive integers, $\mathcal{F}$ the σ-field of all subsets of Ω, and P defined by

$$P(\{n\}) = \frac{1}{n^2} - \frac{1}{(n+1)^2} \quad \text{for all } n = 1, 2, \ldots$$

Let $[n, \infty$ denote the set of all integers $\ge n$ and define

$$\mathcal{F}_n = \sigma(\{1\}, \{2\}, \ldots, \{n\}, [n+1, \infty)), \quad \xi_n = a_n 1_{[n+1,\infty)}, \ a_n > 0,$$

for $n = 1, 2, \ldots$. Find necessary and sufficient conditions on the sequence $\{a_n\}$ for $\{\xi_n, \mathcal{F}_n\}$ to be (i) martingale, (ii) submartingale. In either case, show that ξ_n converges a.s. but not in L^1.

Problem 9. (i) Let η_n, $n \ge 1$, be i.i.d. random variables with $E|\eta_n| < \infty$, θ be another independent random variable with finite mean, and also let $\xi_n = \eta_n + \theta$. Show that $E(\theta|\xi_1, \ldots, \xi_n) \to \theta$ a.s. (ii) Let η_n, $n \ge 1$, be i.i.d. $\mathcal{N}(0, 1)$ random variables, θ be another independent random variable such that $\theta = \pm 1$ with probability $1/2$, and also let $\xi_n = \eta_n \cdot \theta$. What is the a.s. limit of $E(\theta|\xi_1, \ldots, \xi_n)$?