Homework 10 (Stor 635) Last homework!

The due date is April 24 (Thursday), 2008. The homework assignment contains 6 problems concerning martingales, their basic properties, convergence and applications. Individual work is encouraged but you may also discuss the problems with your classmates or me. Good luck!

Problem 1. Exercise 14.10 from Lecture Notes.

Problem 2. If $\{\xi_n, \mathcal{F}_n\}$ and $\{\eta_n, \mathcal{F}_n\}$ are submartingales, show that $\{\xi_n \vee \eta_n, \mathcal{F}_n\}$ is also a submartingale (as usual, $a \vee b = \max\{a, b\}$).

Problem 3. Let ξ_n, $n \geq 1$, be positive i.i.d. random variables with $E\xi_n = 1$ and $P(\xi_n = 1) < 1$. (i) Show that $\eta_n = \prod_{m=1}^n \xi_m$ defines a martingale. (ii) Use the result on a.s. convergence of martingales and further analysis to show that $\eta_n \to 0$ a.s. (iii) Use the strong law of large numbers to conclude that $(1/n)\log \eta_n \to c < 0$ a.s.

Problem 4. Let ξ_n and η_n be nonnegative, integrable and $\mathcal{F}_n$-measurable random variables ($\mathcal{F}_n$ is a sequence of nondecreasing σ-fields). Suppose that

$$E(\xi_{n+1}|\mathcal{F}_n) \leq (1 + \eta_n)\xi_n$$

with $\sum \eta_n < \infty$ a.s. Prove that ξ_n converges a.s. to a finite limit by finding a closely related supermartingale to which a suitable a.s. convergence result could then be applied.

Problem 5. This problem concerns some basic questions about the so-called stopping times. Consider first the following definitions.

A random variable τ taking values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ is called a stopping time if for every $n < \infty$, $\{\tau = n\} \in \mathcal{F}_n$. As usual, $\mathcal{F}_n$ denotes a sequence of nondecreasing σ-fields. A sequence η_n, $n \geq 1$, is said to be predictable if η_n is $\mathcal{F}_{n-1}$-measurable for any $n \geq 1$. For two sequences ξ_n and η_n, define a sequence $\eta \cdot \xi$ by $(\eta \cdot \xi)_n = \sum_{m=1}^n \eta_m(\xi_m - \xi_{m-1})$, $n \geq 1$.

Suppose now that $\mathcal{F}_n = \sigma\{\xi_0, \xi_1, \ldots, \xi_n\}$, where ξ_n is a sequence of random variables. (i) Show that $\{\tau = n\}$ in the definition above can be replaced by $\{\tau \leq n\}$. (ii) The canonical example of a stopping time is $\tau = \inf\{n : \xi_n \in A\}$ for some Borel set A. (By convention, $\inf\{\emptyset\} = \infty$.) Show that τ is indeed a stopping time. (iii) If $\{\xi_n\}_{n\geq 0}$ is a martingale, $\{\eta_n\}_{n\geq 1}$ is a predictable sequence and each η_n is bounded, show that $(\eta \cdot \xi)_n$ is a martingale. (iv) Use (iii) to show that if τ is a stopping time and $\{\xi_n\}$ is a martingale, then $\{\xi_{\tau\wedge n}\}$ is also a martingale (as usual, $a \wedge b = \min\{a, b\}$). Hint: try $\eta_n = 1_{\{n\leq\tau\}}$.

Problem 6. Let η_n, $n \geq 1$, be i.i.d. random variables with $E|\eta_n| < \infty$, θ be another independent random variable with finite mean, and also let $\xi_n = \eta_n + \theta$. For example, if η_n is $\mathcal{N}(0, 1)$, then in statistical terms we have a sample from a normal population with variance 1 and unknown mean. The distribution of θ is called a prior distribution, and $P(\theta \in \cdot\ |\xi_1, \ldots, \xi_n)$ is called the posterior distribution after n observations. Show that $E(\theta|\xi_1, \ldots, \xi_n) \to \theta$ a.s.