Midterm (Stor 635)

March 26, 2008

The midterm contains 11 problems concerning the following topics. Problems 1,2: basic probability. Problem 3: independence. Problem 4: modes of convergence. Problem 5: convergence in distribution. Problem 6: tightness. Problem 7: uniform integrability. Problem 8: Borel-Cantelli lemmas. Problem 9: Kolmogorov 3 series theorem. Problem 10: characteristic functions. Problem 11: CLT. Each problem is worth 3 points. Please write partial solutions or consider special cases. Keep in mind that the midterm is closed book and closed mouth. Good luck!

Problem 1. Provide definitions for the following basic notions of Probability: (*i*) probability space, (*ii*) random variable, (*iii*) distribution function, (*iv*) discrete random variable, (*v*) continuous random variable, (*vi*) singular random variable, (*vii*) expected value, (*viii*) random element.

Problem 2. State and prove Jensen's inequality.

Problem 3. Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent classes of events, each closed under intersections. Let $\mathcal{B}_1 = \sigma(\mathcal{A}_1)$ and $\mathcal{B}_2 = \sigma(\mathcal{A}_2)$. Show that $\mathcal{B}_1$ and $\mathcal{B}_2$ are also independent classes of events.

Problem 4. Compare the following modes of convergence in as much detail as possible: almost sure (a.s.), in probability (in P), in L^p (L^p), and in distribution (in d). In other words, say which mode implies which mode in general and, if you know, provide an idea of proof. If one type does not imply another in general, provide a counterexample and, if you know, state additional assumptions under which the implication holds.

Problem 5. Suppose ξ_n, ξ are absolutely continuous random variables with densities f_n, f. If $f_n \to f$ a.e. (with respect to Lebesgue measure on $\mathbb{R}$), show that $\xi_n \to \xi$ in distribution.

Problem 6. (*i*) Give an example of a family of continuous distribution functions which is not tight. (Show that the considered family, in fact, is not tight.) (*ii*) By Prohorov's theorem, which notion is tightness equivalent to? Define that notion.

Problem 7. Suppose $\xi_n = \mathrm{Exp}(\lambda_n)$, $n \geq 1$, are exponential random variables with parameters $\lambda_n > 0$. (Recall that their densities are $\lambda_n e^{-\lambda_n x}$, $x > 0$.) Find necessary and sufficient conditions for the family $\{\xi_n\}_{n\geq 1}$ to be uniformly integrable.

Problem 8. If $\{\xi_n\}_{n\geq 1}$ are independent, identically distributed random variables, show that

$$E|\xi_1| < \infty \quad \Leftrightarrow \quad \frac{\xi_n}{n} \to 0 \text{ a.s.}$$

Problem 9. Let

$$\xi_n = \begin{cases} \frac{1}{n}, & \text{with prob. } p_n, \\ -\frac{1}{n}, & \text{with prob. } 1 - p_n, \end{cases} \qquad n \geq 1,$$

be independent random variables. Find necessary and sufficient conditions for convergence of the series $\sum_{n=1}^{\infty} \xi_n$ in terms of the sequence $\{p_n\}_{n \geq 1}$.

Problem 10. Let ϕ be a characteristic function of a random variable ξ. (i) If $|\phi(t_0)| = 1$ for some $t_0 > 0$, show that $\xi \in \{a + 2\pi n / t_0,\ n \in \mathbb{Z}\}$ a.s. (ii) If there are $t_k \to 0$ $(t_k > 0)$ such that $|\phi(t_k)| = 1$, show that $\xi = \text{const}$ a.s.

Problem 11. Let

$$\xi_n = \begin{cases} \pm n, & \text{with prob. } \frac{p_n}{2}, \\ 0, & \text{with prob. } 1 - p_n, \end{cases} \qquad n \geq 1,\ p_n > 0,$$

be independent random variables. Give example of $\{p_n\}_{n \geq 1}$ for which:

(i) $\{\xi_n\}_{n \geq 1}$ satisfies the Lindeberg condition,

(ii) $\{\xi_n\}_{n \geq 1}$ does not satisfy the Lindeberg condition.

In case (i), state the resulting convergence according to Central Limit Theorem.