Lecture 1  Pre-“Derivatives” Basics

Stocks and bonds are referred to as underlying basic assets in financial markets. Nowadays, more and more derivatives are constructed and traded whose payoffs depend on the values of these underlying assets or other derivatives. We discuss some issues in this lecture before introducing derivatives.

1.1  Interest rates

1.1.1  Time value of money

“$100 today will be worth more than $100 next year.” It simply means that you can increase the value of $100 by investing it. Albert Einstein once described compound interest as the “greatest mathematical discovery of all time”. The importance of interest rate of a riskless money market lies in its role as a yardstick for measuring the profitability of many risky investment strategies. Suppose stock A and stock B have average annual returns 4% and 10% respectively, while a CD (certificate of deposit) account in your local bank offers 5% annual rate. (You are obligated to keep the account for 5 years, say.) Which one would you choose: A, or B, or CD? It need not be easy to decide due to many factors involved. Still, here is a simplified answer. Using statistical terms, both mean and variance need to be taken into consideration. On the one hand, the CD is better than stock A and worse than stock B in terms of mean profit. On the other hand, greater attention should be paid to variance (usually referred to as volatility in financial jargon). Here the CD is regarded as a bench mark for comparison since it is risk-free, i.e. $100 deposited in the CD will for sure increase to $105 after one year. The actual profitability margin in “stock B vs CD” could be much greater than the mean difference 10% - 5% = 5%, or it is also possible to lose a lot of money in stock B. Both outcomes have positive probabilities to occur. It is this uncertainty that makes investment decisions more challenging. Later we will discuss in more detail the trade-off of “return vs risk”. For one thing, if we want to compare the payoffs of stock B and the CD seriously, we at least need to know the chances for stock B price to go up and down (by a certain amount) in a given time period.

Let \( r_a \) and \( r_m \) denote the annual and monthly interest rates. It is a useful exercise to convert \( r_a \) to an “equivalent” \( r_m \) in the sense that any initial value \( P_0 \) will end up with the same amount, following two different compounding rules (annually and monthly), i.e.

\[
P_0 (1 + r_a) = P_0 (1 + r_m)^{12}. \tag{1.1}
\]

Canceling \( P_0 \) on both sides leads to

\[1 + r_a = (1 + r_m)^{12},\]
hence
\[ r_m = \exp \left[ \frac{\log(1 + r_a)}{12} \right] - 1. \] (1.2)

An approximation rule
\[ r_m = r_a/12 \] (1.3)

is often used. For \( r_a = 6\% \), (1.2) and (1.3) will yield \( r_m = 0.48676\% \) and \( r_m = 0.5\% \) respectively. Another way to interpret (1.2) is that using the same interest rate in a given time period, a more frequent compounding rule will yield a greater return.

1.1.2 An example: mortgage loan

Frank bought a house in 1995 with market value $130,000. He put $30,000 down payment from his saving at the closing and borrowed $100,000 in a 15-year loan with fixed annual (interest) rate 6\%. To figure out his family budget, he calculated the required monthly payment, denoted by \( x \), as follows:

Since 15 years = 180 months, Frank set the equation
\[ x \left[ 1 + (1 + r_m) + (1 + r_m)^2 + \cdots + (1 + r_m)^{179} \right] = 100,000 \left(1 + r_m\right)^{180}. \] (1.4)

Using \( r_m = 0.0048676 \) calculated via (1.2) and the geometric sum formula, Frank had \[ \frac{(1 + r_m)^{180} - 1}{r_m} = 286.91335. \] Therefore,
\[ x = \frac{100,000 \left(1 + 0.0048676\right)^{180}}{286.91335} = 835.2972823. \]

With Frank’s family income, the monthly payment $835.30 is manageable.

**Note:** To justify (1.4), notice that for each month \( n = 1, 2, \ldots, 180 \),
\[ P_{n-1} (1 + r_m) - x = P_n, \]
where \( P_n \) represents the balance at the end of month \( n \). Rewrite this as
\[ x = P_{n-1} (1 + r_m) - P_n \]
then multiplying both sides by \( (1 + r_m)^{180-n} \) yields
\[ x (1 + r_m)^{180-n} = P_{n-1} (1 + r_m)^{180-n+1} - P_n (1 + r_m)^{180-n}. \] (1.5)
Adding all 180 equations (1.5) with $n = 1, 2, ..., 180$ together, and noting that $P_0 = 100,000.00$ and $P_{180} = 0.00$, we will obtain (1.4). (Can you see all intermediate terms on the right hand sides cancel out?)

<table>
<thead>
<tr>
<th>month</th>
<th>beginning balance</th>
<th>monthly payment</th>
<th>monthly interest</th>
<th>principal repayment</th>
<th>ending balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100,000.00</td>
<td>835.30</td>
<td>486.76</td>
<td>348.54</td>
<td>99,651.46</td>
</tr>
<tr>
<td>2</td>
<td>99,651.46</td>
<td>835.30</td>
<td>485.06</td>
<td>350.24</td>
<td>99,301.22</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>96</td>
<td>...</td>
<td>835.30</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>180</td>
<td>...</td>
<td>835.30</td>
<td>...</td>
<td>...</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 1: Amortization schedule for a level-payment fixed-rate mortgage

**Questions:**

(a) How much interest Frank has to pay in addition to the principal $100,000 he borrowed in the mortgage loan?

(b) Suppose having followed this mortgage loan for 8 years, Frank did refinancing in 2003 by converting the original mortgage loan to a 5-year new loan with fixed annual rate 4%. What is Frank’s monthly payment in the new loan?

Frank’s refinancing might be motivated by a number of factors, such as an attractive low interest rate, the need for reducing the monthly payment, etc. More realistic and sophisticated mortgage loans involve both fixed-rate and variable-rate (adjustable rate, floating rate, etc.), such as in the recent sub-prime mortgage loan crisis. Later, we will study the valuation of various fixed-income derivatives in which interest rates are modeled as stochastic processes.

### 1.2 From coin tossing to stock fluctuation

The simplest version of “random walk” — coin tossing — is a probability toy model. In finance literature the term is intended as a cartoon for somewhat unpredictable fluctuations in stock markets. Even the simplest version of random walks can illustrate some behaviors in stock markets. However, it requires a lot more serious thinking and practice to actually fit a (more sophisticated) random walk model by real financial data.

Consider tossing a coin with $P(H) = p$ and $P(T) = q$ (“H = head”, “T = tail”, and the positive constants $p$ and $q$ satisfy $p + q = 1$) $n$ times. We have the following binomial probability formula:
for any integer $k = 0, 1, \ldots, n$,
\[
P(k \text{ H's among } n \text{ tosses}) = \binom{n}{k} p^k q^{n-k}.
\] (1.6)

Now assume an oversimplified stock as follows. In any trading day, the stock price has only two possible moves: up by an amount $\delta$ or down by the same amount $\delta$. The value of $\delta > 0$ is to be specified. Here is a basic mathematical framework:

• $X_t = \text{the increment of the stock price on day } t$, with $P(X_t = \delta) = p$ and $P(X_t = -\delta) = q$.
• $P_0 = \text{the initial stock price at } t = 0$.
• $S_n = X_1 + \cdots + X_n = \sum_{t=1}^{n} X_t = \text{the cumulative increment of the stock price in } n \text{ days}$.
• $P_n = P_0 + S_n = \text{the stock price on day } n$.

It follows immediately from the formula (1.6) that
\[
P(\text{the stock price has } k \text{ up's and } n - k \text{ down's in } n \text{ days}) = \binom{n}{k} p^k q^{n-k}.
\] (1.7)

Plots of this simplified stock price model can be generated by Monte Carlo methods. As an exercise, use Matlab with assigned values $p = q = 1/2$, $\delta = 0.1$ and $n = 1,000$ (trading days in about 4 years). You can try 3 sets of plots with set 1 containing (daily) prices of 1,000 observations, set 2 containing (weekly) prices of 200 observations, and set 3 containing (monthly) prices of 50 observations. Apparently, repeating this experiment by using the same parameters will produce different plots which reflect the uncertainty. Notice the roles played by the parameters $p$, $q$ and $\delta$.

If $p > 1/2$ (or equivalently $p > q$), then the stock has a systematic upward trend, and vice versa. The greater the increment $\delta$, the greater the stock volatility (bigger swing, greater risk, etc.) Trend and volatility are the most important characteristics of a stock. By changing numerical values of $p$, $q$ and $\delta$, we can generate stock plots with different trend and volatility behaviors.

**Question 1:** Do the simple random walk plots really mimic real stock prices?

The answer need not be easy because it depends on how do we tell whether two plots (one for a real stock and the other simulated by a probability model) are “similar”. Some statistical methods provide partial answers. More sophisticated models would be capable of producing plots that resemble many features of real stock markets.

**Question 2:** Are stock markets predictable?

The random walk hypothesis (RWH) or the efficient market hypothesis (EMH) advocated by many economists says virtually no. More specifically, suppose the toy model given here (with
$p = 1/2$) does mimic the typical behavior of a stock (call it stock A). Then we can only say that stock A will go up or down tomorrow with equal chance 50%. Such (noninformative) prediction will remain the same even if we know stock A has been up for the past 10 trading days. In other words, any given information about the prices (or returns) of stock A up to now will not make stock A more or less likely to go up tomorrow. This is due to the independence assumption on stock A increments in different days. However, some financial economists challenged RWH and EMH. See the two books “A Random Walk Down Wall Street” by Burton Malkiel (8th edition, 2003) and “A Non-random Walk Down Wall Street” by Andrew Lo and A. Craig MacKinlay (1999) for the debate. Implications of EMH to asset pricing will also be studied later.

1.3 Return vs risk: mean-variance analysis

1.3.1 Basic concept

A starting point in studying financial markets is to understand return (or reward) — as measured by expected return — and risk — as measured by the variance of the return. For simplicity, we only consider a single period model from time 0 to time 1. Suppose there are $n$ assets in a portfolio. For each $i = 1, \ldots, n$, let $X_i(0)$ and $X_i(1)$ be the time 0 and time 1 values of asset $i$ respectively, and define the return of asset $i$ in the time period to be

$$R_i = \frac{X_i(1) - X_i(0)}{X_i(0)}. \quad (1.8)$$

The portfolio can be represented by $(c_1, \ldots, c_n)$, where $c_i$ is the “proportion” of the initial capital held in asset $i$, $i = 1, \ldots, n$. Note that some $c_i$ could be negative, representing “shorting asset $i$”. The constraint

$$\sum_{i=1}^{n} c_i = 1 \quad (1.9)$$

is often imposed with nonnegative $c_i$. We can define the overall return $R$ for the portfolio in the same manner, and have the relationship

$$R = \sum_{i=1}^{n} c_i R_i. \quad (1.10)$$

Imagine at time 0 an investor is trying to decide what portfolio will yield a high return $R$ at time 1 while reducing the risk. Since $R$ involves the future, it is regarded as a random variable.

1.3.2 Examples

For asset $i$, let $\mu_i = ER_i$ and $\sigma_i^2 = Var(R_i)$ be the expected return (mean) and risk (variance). For assets $i$ and $j$, let $\sigma_{ij} = Cov(R_i, R_j)$ denote the covariance between returns $R_i$ and $R_j$. 

5
Example 1  Assume that all returns are uncorrelated ($\sigma_{ij} = 0$ for all $i$ and $j$) and have the same risk ($\sigma_i^2 = \sigma^2$ for all $i$). If we buy an equal number of shares for all $n$ stocks in the portfolio ($c_i = 1/n$ for all $i$), then $\text{Var}(R) = \sigma^2/n$, i.e. the overall risk for the portfolio is only $1/n$ of the individual asset risk. The more uncorrelated different stocks included in the portfolio, the lower overall risk for the portfolio. This illustrates a principle: “diversification reduces risk”. Furthermore, it is even more helpful in reducing risk if $\sigma_{ij} < 0$ for some $i$ and $j$ (negative correlations). This is a basic idea in hedging.

Example 2  Consider a portfolio consisting of two perfectly correlated assets, i.e. $\text{corr}(R_1, R_2) = 1$ (or equivalently, $\sigma_{12} = \sigma_1 \sigma_2$). Hence

$$\text{Var}(R) = c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2 + 2c_1c_2 \sigma_{12} = (c_1 \sigma_1 + c_2 \sigma_2)^2.$$  

Suppose $\mu_1 = 0.1$, $\sigma_1 = 0.08$, $\mu_2 = 0.14$, and $\sigma_2 = 0.18$. We can set $c_1 = 1.8$ and $c_2 = -0.8$, which means that for every $100$ invested, we long asset $1$ with $180$ and short asset $2$ with $80$. The expected return for the portfolio will be

$$\text{ER} = 1.8 \cdot 0.1 + (-0.8) \cdot 0.14 = 0.068.$$  

But more importantly,

$$\text{Var}(R) = 0,$$

which implies that the portfolio is risk-free. In other words, to avoid an arbitrage opportunity (“free lunch”), any available risk-free investment (such as a bank account) has to have a return rate no lower than $0.068$.

There is a trade-off between raising the expected return $\text{ER}$ and reducing the risk $\text{Var}(R)$ for a portfolio. In general, you cannot achieve both at the same time. Here is a portfolio optimization problem: Suppose we know $\mu_i, \sigma_i$ and $\sigma_{ij}$ for all $i, j = 1, \ldots, n$. Given the mean return $\mu = \text{ER}$, construct a portfolio $(c_1, \ldots, c_n)$ such that its risk

$$\sigma^2 = \text{Var}(R) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_ic_j \sigma_{ij}$$

is minimized.

For $n \geq 3$, this constrained quadratic optimization problem can be solved by using calculus. In practice, the parameters $\mu_i$, $\sigma_i$, $\sigma_{ij}$, $i, j = 1, \ldots, n$ need not be given. They should be estimated based on financial data. This will be in a forthcoming homework assignment.