11.1 Reflection principle and the first passage time

Define $M_t = \max_{0 \leq u \leq t} W_u$ and recall the first passage time $\tau(a) = \inf\{t \geq 0 : W_t = a\}$.

**Proposition 11.1** For every $a > 0$,

$$P(M_t \geq a) = P(\tau(a) < t) = 2 \ P(W_t > a) = 2 - 2 \Phi(a/\sqrt{t}),$$  \hfill (11.1)

where $\Phi(\cdot)$ denotes the cdf of standard normal distribution.

**Note:** The rigorous proof relies on the strong Markov property of $W$, but drawing a picture would provide a heuristic argument. Furthermore, Theorem 10.1 implies that $W^*$ is also a Brownian motion with $W^*_t = W_{\tau(a)+t} - W_{\tau(a)}$.

11.2 Behaviors of sample paths

Although being continuous, Brownian paths have "crazy" behaviors.

11.2.1 Zero set

**Proposition 11.2** Let $S = \{t \geq 0 : W_t = 0\}$. Then with probability one,

(i) $\text{meas}(S) = 0$, where "meas" denotes the Lebesgue measure;

(ii) in every interval $(t_1, t_2) \subset \mathbb{R}_+$, there are infinitely many elements of $S$;

(iii) $S$ is a perfect set, i.e. $S$ is closed and dense in itself.

**Note:** To show (i), it follows from Fubini’s Theorem that

$$E[\text{meas}(S)] = E \int_0^\infty I_{\{W_t = 0\}} \ dt = \int_0^\infty P(W_t = 0) \ dt = 0.$$

Proposition 11.1 helps us understand why (ii) holds.
11.2.2 Non-differentiability

Proposition 11.3 For every $t \geq 0$, the Brownian path is almost surely non-differentiable at $t$.

Note: The set of probability zero depends on $t$. It follows immediately that, with probability one, the Brownian path is non-differentiable at any rational point. The following stronger result is due to Dvoretsky, Erdős and Kakutani:

Theorem 11.1 With probability one, the Brownian path is nowhere differentiable.

It is obvious that any “picture” of a Brownian path would doom to be wrong qualitatively.

11.2.3 Quadratic variation

For a stochastic process $X = \{X_t\}_{t \geq 0}$ and $p > 0$, define the $p$-th variation process of $X$ by

$$
\langle X, X \rangle_t^{(p)} \triangleq \lim_{\|\Pi\| \to 0} \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|^p
$$

if the limit exists, where $\|\Pi\| = \max\{|t_k - t_{k-1}| : k = 1, ..., n\}$ for the partition $\Pi : 0 = t_0 < t_1 < \cdots < t_n = t$ of $[0, t]$. The cases $p = 1$ and $p = 2$ are referred to as the total variation and quadratic variation respectively. In particular, when $p = 2$, we simply suppress the superscript and write $\langle X, X \rangle_t \triangleq \langle X, X \rangle_t^{(2)}$.

Theorem 11.2 With probability one, the quadratic variation process of Brownian motion satisfies

$$
\langle W, W \rangle_t = t.
$$

Exercise 11.1 Show that for Brownian motion, the total variation $\langle W, W \rangle_t^{(1)} = \infty$ with probability one.

Theorem 11.1 has an important impact on Itô’s stochastic calculus, in particular on the famous Itô’s formula to be presented later.