Lecture 12  Stochastic Integration

12.1  The Itô integral of simple processes

We will define the Itô stochastic integral in this lecture through an approximation procedure, same as in the definition of the Riemann integral. Let $W$ be a standard Brownian motion and $\{F_t\}_{t \geq 0}$ be the Brownian filtration. An adapted process $h = \{h_t\}$ is said to be simple if it is a random step function:

$$h_t = \sum_{i=1}^{k} \xi_i I(t_i, t_{i+1}) (t), \quad (12.1)$$

for some positive integer $k$, a finite sequence $0 < t_1 < \cdots < t_{k+1} \leq T$, and random variables $\xi_1, \ldots, \xi_k$ such that $\xi_i$ is $F_{t_i}$-measurable.

**Definition 12.1**  For a simple process $h$, define the Itô integral

$$I(h) = \int_0^T h_t \, dW_t = \sum_{i=1}^{k} \xi_i (W_{t_{i+1}} - W_{t_i}). \quad (12.2)$$

**Proposition 12.1**  $I(h)$ satisfies the following properties:

$$EI(h) = 0 \quad (12.3)$$

$$E[I(h)]^2 = \int_0^T Eh_t^2 \, dt \quad (12.4)$$

$$\int_0^T (ah_t + bh'_t) \, dW_t = a \int_0^T h_t \, dW_t + b \int_0^T h'_t \, dW_t \quad (12.5)$$

for any constants $a, b$ and any simple processes $h$ and $h'$.

$I(h)$ defined in (12.2) is $F_{t_{k+1}}$-measurable.

**Note:** It is a good exercise to show (12.3) and (12.4). In particular, you will see why is crucial to have independence between $\xi_i$ and $W_{t_{i+1}} - W_{t_i}$ in each term of (12.2).

12.2  The Itô integral of $H^2$ processes

An adapted process $h$ is said to be a member of $H^2$ if $\int_0^T Eh_t^2 \, dt < \infty$. 

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Proposition 12.2 For $h \in H^2$, there exists a sequence of simple processes $h^{(n)}$ such that

$$\lim_{n \to \infty} \int_0^T E[h^{(n)}_t - h_t]^2 \, dt = 0. \quad (12.6)$$

Definition 12.2 For $h \in H^2$ and any approximating sequence $\{h^{(n)}\}$ as in (12.6), define

$$I(h) = \int_0^T h_t \, dW_t = \lim_{n \to \infty} \int_0^T h^{(n)}_t \, dW_t = \lim_{n \to \infty} I(h^{(n)}). \quad (12.7)$$

Note: It can be shown that the limit in (12.7) exists in $L^2$ sense and does not depend on the approximating sequence $\{h^{(n)}\}$.

Theorem 12.1 Properties (12.3), (12.4) and (12.5) hold for $H^2$ processes. Furthermore, let

$$I_t = \int_0^t h_u \, dW_u, \quad 0 \leq t \leq T.$$  

Then the process $I = \{I_t\}$ is an adapted martingale; with probability one, $I$ has a continuous sample path and the quadratic variation process

$$\langle I, I \rangle_t = \int_0^t h_u^2 \, du. \quad (12.8)$$

Note: Definition of the Itô integral can be extended to a broader class of processes $h$ that satisfy $P(\int_0^T h_t^2 \, dt < \infty) = 1$ by using “local martingales”. We will skip that part.

Example 12.1 We are to compute $\int_0^t W_u \, dW_u$ using Definition 12.2. In practice, this is hardly the method of choice for any problem, due to the useful Itô formula which will be presented in the next lecture. Nevertheless, the calculation here demonstrates the role played by the quadratic variation of Brownian motion and a qualitative difference between the Fundamental Theorem of Calculus and the Itô stochastic calculus.

For a positive integer $n$, define a simple process $h^{(n)}$:

$$h^{(n)}_u = W_{t_{i-1}}, \quad t_{i-1} \leq u < t_i,$$

where $t_i = it/n$, $i = 1, \ldots, n$. First, it can be shown that

$$\lim_{n \to \infty} \int_0^t E[h^{(n)}_u - W_u]^2 \, du = 0 \quad (12.9)$$

(done in class). Second, we have

$$\frac{1}{2} \sum_{i=1}^n (W_{t_i} - W_{t_{i-1}})^2 = \frac{1}{2} W_t^2 - \sum_{i=1}^n W_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}). \quad (12.10)$$
(Exercise!) Letting \( n \to \infty \), the left-hand side of (12.10) tends to \( t/2 \) due to the quadratic variation of Brownian motion, and the right-hand side tends to \( W_t^2/2 - \int_0^t W_u \, dW_u \). This yields

\[
\int_0^t W_u \, dW_u = \frac{W_t^2}{2} - \frac{t}{2}.
\]  

(12.11)

Note that (12.11) differs from the Fundamental Theorem of Calculus due to the extra term \(-t/2\) on the right-hand side.