Itô’s formula is at the heart of stochastic calculus. It shows an important distinction from the Fundamental Theorem of Calculus. We will present the 1D and multidimensional versions of Itô’s formula and illustrate their applications in a number of examples.

13.1 The 1D case

**Definition 13.1** $X = \{X_t\}$ is called a 1D Itô process if it has an expression

$$X_t = X_0 + \int_0^t \mu_u \, du + \int_0^t \sigma_u \, dW_u, \quad 0 \leq t \leq T,$$

(13.1)

where the drift process $\mu = \{\mu_t\}$ is adapted to the Brownian filtration, and the diffusion coefficient $\sigma = \{\sigma_t\}$ is a $H^2$-process (see Lecture 12).

Sometimes an Itô process $X$ is defined as a solution to the stochastic differential equation (SDE)

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t.$$  

(13.2)

Note that (13.2) is not well-defined in a strict sense because Brownian motion is non-differentiable. From now on, whenever we see a differential form (13.2), it should be understood as defined by (13.1).

**Theorem 13.1** Let $g(t, x)$ be continuously differentiable in $t$ and twice continuously differentiable in $x$. Define $Y_t = g(t, X_t)$, $0 \leq t \leq T$. Then $Y = \{Y_t\}$ is an Itô process that satisfies

$$dY_t = \frac{\partial g(t, X_t)}{\partial t} \, dt + \frac{\partial g(t, X_t)}{\partial x} \, dX_t + \frac{1}{2} \frac{\partial^2 g(t, X_t)}{\partial x^2} \, (dX_t)^2$$

(13.3)

Note:

(a) The following rules are used in the expansion of $(dX_t)^2 = d\langle X, X \rangle_t$: $(dt)^2 = (dt)(dW_t) = 0$ (higher-order infinitesimals); $(dW_t)^2 = d\langle W, W \rangle_t = dt$ (quadratic variation of Brownian motion).

(b) The Itô’s formula (13.3) can be expressed as an integral form

$$Y_t = Y_0 + \int_0^t \left[ \frac{\partial g(u, X_u)}{\partial u} \, \mu_u + \frac{\partial g(u, X_u)}{\partial u} \, \mu_u + \frac{1}{2} \frac{\partial^2 g(u, X_u)}{\partial x^2} \, \sigma_u \right] \, du$$

$$+ \int_0^t \frac{\partial g(u, X_u)}{\partial x} \, \sigma_u \, dW_u.$$  

(13.4)
Theorem 13.1 is proved by Taylor expansion and estimation of the remainder. See the book by Karatzas and Shreve for details.

13.2 The multidimensional case

For a positive integer \( m \), \( W = (W^{(1)}, ..., W^{(m)})' \) is called a \( m \)-dimensional Brownian motion if \( W^{(1)}, ..., W^{(m)} \) are independent 1D standard Brownian motions. Here \( A' \) denotes the transpose of matrix \( A \). Let \( \{\mathcal{F}_t\}_{t \geq 0} \) be the Brownian filtration generated by \( W \).

\[ \text{Definition 13.2} \quad X = (X^{(1)}, ..., X^{(k)})' \text{ is called a } k\text{-dimensional Itô process if it has an expression} \]

\[ dX_t = \mu_t \, dt + \sigma_t \, dW_t, \quad 0 \leq t \leq T, \tag{13.5} \]

where \( W = \{W_t\} \) is a \( m \)-dimensional Brownian motion, the (vector) drift process \( \mu = \{\mu_t\} = (\mu^{(1)}, ..., \mu^{(k)})' \) is adapted to the Brownian filtration, the diffusion coefficient \( \sigma = \{\sigma_t\} \) is a \( k \times m \) matrix

\[ \sigma_t = \begin{pmatrix} \sigma_t^{(1)} & \cdots & \sigma_t^{(m)} \\ \vdots & \ddots & \vdots \\ \sigma_t^{(k)} & \cdots & \sigma_t^{(km)} \end{pmatrix}, \tag{13.6} \]

where all entries \( \sigma^{(1)}, ..., \sigma^{(km)} \) are \( H^2 \)-processes.

Theorem 13.2 Let \( g(t, x) \), \( x = (x_1, ..., x_k) \) be continuously differentiable in \( t \) and twice continuously differentiable in \( x \in \mathbb{R}^k \). Define \( Y_t = g(t, X_t), 0 \leq t \leq T \). Then \( Y = \{Y_t\} \) is an Itô process that satisfies

\[ dY_t = \frac{\partial g(t, X_t)}{\partial t} \, dt + \sum_{i=1}^k \frac{\partial g(t, X_t)}{\partial x_i} \, dX_t^{(i)} + \sum_{i,j=1}^k \frac{1}{2} \frac{\partial^2 g(t, X_t)}{\partial x_i \partial x_j} \, dX_t^{(i)} dX_t^{(j)}, \tag{13.7} \]

where the following rules are adopted in the expansion of \( dX_t^{(i)} dX_t^{(j)} \): \( (dt)^2 = (dt)(dW_t^{(i)}) = 0 \) (higher-order infinitesimals); \( (dW_t^{(i)})^2 = dt \) (quadratic variation of Brownian motion); \( dW_t^{(i)} dW_t^{(j)} = 0 \) for \( i \neq j \) (independence).

The following Itô product rule follows from Theorem 13.2.

Corollary 13.1 Let \( X, Y \) be two 1D Itô processes. Then

\[ d(X_t Y_t) = X_t \, dY_t + Y_t \, dX_t + dX_t \, dY_t. \tag{13.8} \]
13.3 Examples

Example 13.1 Verify \( \int_0^t W_u \, dW_u = W_t^2/2 - t/2 \).

Example 13.2 (geometric Brownian motion) Show that \( S_t = S_0 \exp[(\mu - \sigma^2/2) t + \sigma W_t] \) satisfies the SDE \( dS_t/S_t = \mu \, dt + \sigma \, dW_t \) with constants \( \mu \in \mathbb{R} \) and \( \sigma > 0 \).

Example 13.3 (Ornstein-Uhlenbeck process) Show that \( X_t = X_0 e^{-\mu t} + \sigma \int_0^t e^{-\mu (t-u)} \, dW_u \) satisfies the SDE \( dX_t = -\mu X_t \, dt + \sigma \, dW_t \) with positive constants \( \mu \) and \( \sigma \).

Example 13.4 Consider the SDE

\[
dX_t = (A_t X_t + \mu_t) \, dt + \sum_{i=1}^m [B_t^{(i)} X_t + \sigma_t^{(i)}] \, dW_t^{(i)}, \quad 0 \leq t \leq T, \tag{13.9}
\]

where \( W = (W^{(1)}, ..., W^{(m)}) \) is a \( m \)-dimensional Brownian motion, and \( \{A_t\}, \{\mu_t\}, \{B_t^{(i)}\}, \{\sigma_t^{(i)}\}, i = 1, ..., m \) are adapted processes. Use Itô’s formula to show the following \( X_t \) satisfies (13.9):

\[
X_t = \xi_t \left[ X_0 + \int_0^t \xi_u^{-1} \left( \mu_u - \sum_{i=1}^m B_u^{(i)} \sigma_u^{(i)} \right) \, du + \sum_{i=1}^m \int_0^t \xi_u^{-1} \sigma_u^{(i)} \, dW_u^{(i)} \right], \tag{13.10}
\]

where

\[
\xi_t = \exp \left[ \int_0^t A_u \, du + \sum_{i=1}^m \int_0^t B_u^{(i)} \, dW_u^{(i)} - \frac{1}{2} \sum_{i=1}^m \int_0^t (B_u^{(i)})^2 \, du \right]
\].

Example 13.5 Use Itô’s formula to calculate \( EY_t \) where \( Y_t = e^{X_t} \int_0^t e^{-X_u} \, dW_u \) and \( X_t = at + bW_t \) with constants \( a \in \mathbb{R} \) and \( b > 0 \).