Lecture 15  Girsanov Theorem and Risk Neutral Valuation

Assume the basic setting in a BS market: a riskless asset $B$ following (14.1), a risky asset $S$ following (14.2) with the Brownian filtration $\{\mathcal{F}_t\}$, defined in a probability space $(\Omega, \mathcal{F}, P)$.

15.1 Value processes, self-financing strategies, arbitrage

An adapted process $h = \{(h_{0t}, h_{1t}) : 0 \leq t \leq T\}$ is called a dynamic portfolio in which $h_{0t}B_t$ represents the balance of bank account $B$ at time $t$ and $h_{1t}$ is the number of shares of stock $S$ at time $t$. In general, we should define predictability in continuous-time and require $h$ to be a predictable process. However, it is not necessary here due to the continuous sample paths of Brownian motion $W$, i.e. if $h$ is adapted to the Brownian filtration, then it is predictable.

$V = \{V_t\}$ is called the value process of a portfolio $h$ where

$$V_t = h_{0t}B_t + h_{1t}S_t, \quad t \in [0, T]. \quad (15.1)$$

It is intuitive to call $h$ a self-financing strategy if

$$dV_t = h_{0t} dB_t + h_{1t} dS_t \quad (15.2)$$

at every $t$. But a formal definition is required.

**Definition 15.1** Assume $\int_0^T E|h_{0t}| \, dt + \int_0^T Eh_{1t}^2 \, dt < \infty$. $h$ is called a self-financing strategy if

$$V_t = V_0 + \int_0^t h_{0u} dB_u + \int_0^t h_{1u} dS_u \quad (15.3)$$

for $0 < t \leq T$ with probability one.

**Note:**

(i) It follows from (14.1) and (14.2) that

$$\int_0^t h_{0u} dB_u = \int_0^t h_{0u} re^{ru} \, du$$

and

$$\int_0^t h_{1u} dS_u = \int_0^t h_{1u} S_u \mu \, du + \int_0^t h_{1u} S_u \sigma dW_u$$

for $0 < t \leq T$. 

(ii) Define the discounted value \( V_t^* = V_t / B_t \). Then the self-financing condition can be given by

\[
V_t^* = V_0 + \int_0^t h_{1u} dS_u^*
\]

for \( 0 < t \leq T \) with probability one. In fact,

\[
dV_t^* = -r V_t^* dt + e^{-rt} dV_t
\]

\[
= -re^{-rt}(h_{0u} e^{rt} + h_{1u} S_t) dt + e^{-rt} h_{0u} de^{rt} + e^{-rt} h_{1u} dS_t
\]

\[
= h_{1t} (-re^{-rt} S_t dt + e^{-rt} dS_t)
\]

\[
= h_{1t} dS_t^*.
\]

**Definition 15.2** An arbitrage opportunity is said to exist if there is a self-financing strategy \( h \) such that its value process satisfies \( V_0 = 0 \), but \( P(V_t \geq 0) = 1 \) and \( P(V_t > 0) > 0 \) for some \( t \in (0, T] \).

We will define risk neutral probability measures [equivalent martingale measures (EMM)] and show that the existence of a risk neutral measure is a sufficient condition for no arbitrage.

### 15.2 Girsanov Theorem for a single Brownian motion

Recall that a probability measure \( Q \) on \( (\Omega, \mathcal{F}) \) is said to be absolutely continuous with respect to \( P \), denoted by \( Q \ll P \), if any \( A \in \mathcal{F} \) with \( P(A) = 0 \) would imply \( Q(A) = 0 \). It is well-known in probability theory that \( Q \ll P \) if and only if there exists a non-negative random variable \( Z \) such that \( Q(A) = \int_A Z dP \) for all \( A \in \mathcal{F} \). \( Z \) is called the Radon-Nikodým derivative (or density) of \( Q \) with respect to \( P \), written as \( Z = dQ/dP \). Moreover, \( P \) and \( Q \) are said to equivalent, denoted by \( P \sim Q \), if both \( Q \ll P \) and \( P \ll Q \) hold. Note that \( P \sim Q \) if and only if \( P(Z > 0) = 1 \).

**Theorem 15.1** Let \( \{\theta_t\} \) be an adapted process satisfying \( E[\exp(\int_0^T \theta_t^2 dt)/2] < \infty \) (Novikov condition). For \( 0 \leq t \leq T \), define

\[
Z_t = \exp \left( -\int_0^t \theta_u dW_u - \frac{1}{2} \int_0^t \theta_u^2 du \right), \quad (15.5)
\]

\[
\tilde{W}_t = W_t + \int_0^t \theta_u du, \quad (15.6)
\]

and a measure \( Q \) with \( dQ/dP = Z_T \). Then \( \{Z_t\} \) is a martingale under \( P \) and \( \{\tilde{W}_t\} \) is a standard Brownian motion under \( Q \).

The following martingale representation theorem is a basis for marketability of contingent claims in the BS market.
Theorem 15.2 For every square-integrable martingale \( X = \{X_t\} \in R_+ \) (\( EX_t^2 < \infty \) for every \( t \)) with respect to the Brownian filtration, there is an adapted process \( H \) such that \( \int_0^T EH_t^2 \, dt < \infty \), and
\[
X_t = X_0 + \int_0^t H_u \, dW_u, \quad t \in [0, T].
\]
(15.7)

See Karatzas and Shreve for the proofs of Theorem 15.1 and Theorem 15.2.

Corollary 15.1 For a \( \mathcal{F}_T \)-measurable random variable \( Y \) with \( EY^2 < \infty \), there is an adapted process \( H \) such that \( \int_0^T EH_t^2 \, dt < \infty \), and
\[
Y = EY + \int_0^t H_u \, dW_u, \quad t \in [0, T].
\]
(15.8)

15.3 Risk neutral valuation in the BS market

A contingent claim \( Y \), defined as a \( \mathcal{F}_T \)-measurable non-negative random variable with \( EY^2 < \infty \), is said to be marketable if there is a self-financing dynamic portfolio \( h \) such that the value process \( V \) satisfies \( V_T = Y \). The following result is a risk neutral valuation principle in the BS market.

Theorem 15.3 In the BS market, every contingent claim \( Y \) is replicable by a portfolio \( h \) with the time-\( t \) value
\[
V_t = E_Q \left[ e^{-r(T-t)}Y \mid \mathcal{F}_t \right],
\]
where \( Q \) is the risk-neutral measure defined by letting \( \theta_t = \frac{\mu - r}{\sigma} \forall t \in [0, T] \) in Theorem 15.1.

Proof: Suppose \( Y \) is replicable by \( h \) with \( V_T = Y \). It follows from (14.10) or (14.11) that setting \( \theta_t = \frac{\mu - r}{\sigma} \forall t \in [0, T] \) in Theorem 15.1 will make \( S^* \) a \( Q \)-martingale. By (15.4), \( V^* \) will be a \( Q \)-martingale, which with \( V_T = Y \) will imply (15.9). To show \( Y \) is marketable, note that \( X_t = E_Q(e^{-rT}Y | \mathcal{F}_t) \) forms a \( Q \)-martingale, Theorem 15.2 implies that
\[
X_t = X_0 + \int_0^t H_u \, dW_u
\]
(15.10)
for some adapted process \( \{H_t\} \). Letting
\[
h_{1t} = H_t / (\sigma S_t^*) \quad \text{and} \quad h_{0t} = X_t - h_{1t} S_t^*
\]
(15.11)
will define a self-financing strategy with the value process
\[
V_t = e^{rt}X_t = E_Q \left[ e^{-r(T-t)}Y \mid \mathcal{F}_t \right],
\]
(15.12)
and obviously \( V_T = Y \).

The discussion in this lecture indicates that the BS market is a continuous-time analogue of the binomial tree model. We will study the Fundamental Theorems of Asset Pricing later.