Consider a special case of Theorem 15.3 with a contingent claim \( Y = g(S_T) \) — a European style derivative written on the stock price \( S_T \). Denote its time \( t \) value by \( P(t, S_t) \), and the partial derivatives by \( \frac{\partial P(t, S_t)}{\partial t} = P_t, \frac{\partial P(t, S_t)}{\partial x} = P_x, \frac{\partial^2 P(t, S_t)}{\partial x^2} = P_{xx} \) respectively. Itô’s formula yields

\[
e^{rt} d[B_t^{-1} P(t, S_t)] = P_t dt - rP(t, S_t) dt + P_x dS_t + \frac{1}{2} P_{xx} (dS_t)^2
\]

in which the risk neutral dynamics \( dS_t = rS_t dt + \sigma S_t dW_t \) is adopted. Since \( B_t^{-1} P(t, S_t) \) is a \( Q \)-martingale, the coefficient of \( dt \) in the drift term has to be zero. Therefore, \( P(t, S_t) \) must satisfy the BS PDE

\[
\frac{\partial P(t, x)}{\partial t} + rx \frac{\partial P(t, x)}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 P(t, x)}{\partial x^2} - r P(t, x) = 0 \tag{16.1}
\]

with the terminal condition \( P(T, x) = g(x) \).

Note:

(i) Itô’s formula only applies to those functions \( P(t, x) \) that satisfy certain differentiability conditions. In general, we do not know whether that is the case in advance. Of course we do know, based on (14.7) — (14.9), the call option price \( P(t, S_t) \triangleq C_{BS}(S_t, \sigma^2, r, K, T - t) = S_t \Phi(d_1) - e^{-r(T-t)} K \Phi(d_2) \) is indeed a smooth function with respect to \( t \) and \( S_t = x \). In this sense, the argument here only shows that the BS PDE verifies the solution for a derivative price. Furthermore, under certain conditions (what?) on the function \( g \), we would be able to show that the function \( P(t, x) = E_Q \left[ e^{-r(T-t)} g(S_T) | S_t = x \right] \) is twice differentiable in \( x \).

(ii) The purpose for introducing the BS PDE is to provide an alternative analytical approach for derivation of a pricing formula, independent of the probabilistic risk neutral valuation principle via a conditional expectation under \( Q \). In fact, such an approach is applicable to an extensive class of problems. Here is a general strategy: establish a parabolic PDE with a certain boundary condition by differentiating \( P(t, x) \), show that the boundary value problem has a unique solution, then solve it analytically (when you are lucky) or numerically (more often) by using a finite difference method, etc.

This approach can be extended to a more general setting, which we still call a BS market, without too much effort.

Theorem 16.1 Assume a bank account process \( B_t = \exp(\int_0^t r_u du) \) with a deterministic interest rate function \( r_t \). A risky asset process satisfies the SDE

\[
dS_t = \mu_t(S_t) dt + \sigma_t(S_t) dW_t \tag{16.2}
\]
under a physical measure $P$. Let

$$P(t, x) = E_Q \left[ \exp \left( - \int_t^T r_s du \right) g(S_T) | S_t = x \right]$$

(16.3)

where $Q$ is a risk neutral measure obtained from $P$ through the Girsanov transformation with $\theta_t = (\mu_t - r_t)/\sigma_t$. Under certain regularity conditions on functions $\mu_t, \sigma_t, r_t$ and $g$, $P(t, x)$ will satisfy the PDE

$$\frac{\partial P(t, x)}{\partial t} + r_t \frac{\partial P(t, x)}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 P(t, x)}{\partial x^2} - r_t P(t, x) = 0$$

(16.4)

with the terminal condition $P(T, x) = g(x)$. Conversely, the boundary value problem has a unique solution that has the representation (16.3).

Theorem 16.1 can be proved using the same argument in Lecture 15 and Lecture 16. First, we verify that $B_t^{-1} P(t, S_t)$ is a $Q$-martingale. Second, applying Itô’s formula to $B_t^{-1} P(t, S_t)$ will yield (16.4). Finally, the converse statement would rely on the standard PDE theory regarding the existence and uniqueness of solutions (we skip it). Theorem 16.1 is a special case and an important application of the famous Feynman-Kac formula in stochastic calculus. A more general version of Feynman-Kac formula is available to fit various applications in which the Brownian shocks are multidimensional, interest rates are stochastic, etc.