American option pricing in continuous-time is based on the same idea as in discrete-time, i.e. formulated as an optimal stopping problem. However, the technicalities involved are much more complicated. The introduction we give in this lecture follows the book *Introduction to Stochastic Calculus Applied to Finance* by Lamberton and Lapeyre (2000, CRC Press, Section 4.4). Another good reference is the book *Martingale Methods in Financial Modeling* by Musiela and Rutkowski (1997, Springer, Chapter 8).

### 17.1 Minimal value and optimal stopping

Consider the basic framework in Section 15.1 except for an additional consumption process $C_t \triangleq \{C_t\}$ with $C_0 = 0$, which is adapted, non-decreasing in $t$ and has continuous sample paths with probability one. $C_t$ represents the cumulative consumption up to time $t$. Adding the consumption process $C$ will make the arbitrage valuation of American contingent claims convenient (although not necessary). However, we may not have an opportunity to appreciate such a modification since most of the technical arguments will be skipped.

**Definition 17.1** Assume $\int_0^T E|h_0| \, dt + \int_0^T E|h_1|^2 \, dt < \infty$. $(h, C)$ is called a self-financing trading and consumption strategy if the value process satisfies

$$V_t = V_0 + \int_0^t h_{0u} \, dB_u + \int_0^t h_{1u} \, dS_u - C_t$$

for $0 < t \leq T$ with probability one.

At each $t \in [0, T]$, the holder of an American option can choose between “go on” and “stop and take an immediate payoff $Y_t$”. We only consider the special case with a payoff process $Y_t = g(S_t)$, where $g$ is a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$. Denote the value process of an American option with the payoff function $g$ by $V(g) = \{V_t(g)\}_{t \in [0, T]}$, where $V_t(g) = h_{0u} B_t + h_{11} S_t$.

**Definition 17.2** $h$ is said to hedge an American option with payoff $g$ if with probability one, we have $V_t(h) \geq g(S_t)$, $\forall t \in [0, T]$. Denote by $H(g)$ the set of trading strategies with consumption that hedge an American option with payoff $g$.

**Theorem 17.1** Define

$$v_t(x) = \sup_{\tau} \mathbb{E}_Q \left[ e^{-r(\tau-t)} g(x, \xi_{t, \tau}) \right],$$

where $Q$ is a risk neutral measure with $\theta_t \equiv (\mu - r)/\sigma$, $\xi_{t, \tau} = \exp[(r - \sigma^2/2)(\tau-t) + \sigma (W_\tau-W_t)]$ and $\tau$ is a stopping time taking values in $[t, T]$. Then there exists a strategy $h^*$ such that $V_t(h^*) = v_t(S_t)$,
\( \forall t \in [0, T] \). Moreover, \( v_t(S_t) \leq V_t(h), \forall t \in [0, T] \) and any strategy \( h \in H(g) \). At time \( t \) with state \( S_t \), \( v_t(S_t) \) can be interpreted as the minimal value of a hedging scheme for an American option. The American option should be exercised at

\[
\tau_t(S_t) = \inf \{ u \in [0, T] : v_u(S_u) = g(S_u) \} \tag{17.3}
\]

**Note:** See Karatzas and Shreve for the proof. The process \( \{e^{-rt}v_t(S_t)\} \) is the Snell envelope of \( \{e^{-rt}g(S_t)\} \) in the sense it is the smallest \( Q \)-supermartingale that bounds it from above. Therefore, \( v_t(S_t) \) should be considered as the time-\( t \) price of the American option.

It turns out that an American call is the same as an European call when no dividends are paid on the stock.

**Proposition 17.1** For \( g(S_t) = (S_t - K)^+ \) at every \( t \in [0, T] \), we have

\[
v_t(S_t) = C^{BS}(S_t, \sigma^2, r, K, T-t), \text{ which is the time-} t \text{ price of an European call.}
\]

**Proof** Without loss of generality, set \( t = 0 \). It suffices to show that for any stopping time \( \tau \leq T \),

\[
E_Q[e^{-r\tau}(S_{\tau} - K)^+] \leq E_Q[e^{-rT}(S_T - K)^+] = E_Q(S_T^* - e^{-rT}K)^+.
\]

That \( S^* \) is a \( Q \)-martingale implies

\[
E_Q[(S_T^* - e^{-rT}K)^+|\mathcal{F}_\tau] \geq E_Q[(S_T^* - e^{-rT}K)|\mathcal{F}_\tau] = S_T^* - e^{-rT}K \geq S_\tau^* - e^{-r\tau}K,
\]

Hence \( E_Q[(S_T^* - e^{-rT}K)^+|\mathcal{F}_\tau] \geq (S_\tau^* - e^{-r\tau}K)^+ \), and taking \( E_Q(\cdot) \) on both sides will yield (17.4).

### 17.2 Perpetual American put

Set \( g(S_t) = (K - S_t)^+ \). A perpetual American put is the simplest American put that can be exercised at any large time. Although it is not traded in the market, it enjoys an explicit solution hence helps us get familiar with the calculation. Pricing American puts in general has to resort to numerical computation.

To ease the notation, we set \( t = 0 \) (this can be done by replacing \( T \) with \( T - t \)). (17.2) is reduced to

\[
v_0(S_0) = \sup_{\tau} E_Q[Ke^{-r\tau} - S_0 \exp(\sigma W_\tau - \sigma^2\tau/2)]^+ \tag{17.5}
\]

where \( \tau \in [0, T] \) is a stopping time. If we drop the assumption of fixed finite horizon \( T \) and the restriction \( \tau \leq T \), then

\[
v_0(S_0) = \sup_{\tau} E_Q\{[Ke^{-r\tau} - S_0 \exp(\sigma W_\tau - \sigma^2\tau/2)]^+I_{\{\tau < \infty\}}\} \tag{17.6}
\]
Proposition 17.2  The time-0 value of a perpetual American put is given by

\[ v_0(S_0) = \begin{cases} 
K - S_0, & \text{for } S_0 \leq a^* \\
(K - a^*)(S_0/a^*)^{-b}, & \text{for } S_0 > a^*
\end{cases} \]

(17.7)

where \( a^* = K b / (1 + b) \) and \( b = 2r/\sigma^2 \).

See Lamberton and Lapeyre for the proof.

Note: Returning to an American put with finite maturity \( T \), similar argument in proving Proposition 17.2 implies that for every \( t \in [0, T] \), there exists a threshold \( a_t \) such that \( v_t(S_t) = K - S_t \) \( \forall S_t \leq a_t \) and \( v_t(S_t) > (K - S_t)^+ \forall S_t > a_t \). Moreover, \( a_t \geq a^* \forall t \in [0, T] \). This shows that the optimal exercise strategy for an American put is a “threshold rule” based on the value \( S_t \). However, no explicit expression for \( a_t \) is available.