Lecture 19 Interest Rate Models

19.1 Basic elements in a bond market

Consider a market money account $B = \{B_t\}$ with $B_0 = 1$, $B_t = \exp(\int_0^t r_u du)$, $t \in [0, T]$, where the short rate $\{r_t\}$ is assumed to be a stochastic process. The bank account $B$ and related term structures ($\{B(t, \tau)\}$, or $\{Y(t, \tau)\}$, or $\{f(t, \tau)\}$) in a bond market serve as underlying assets similar to stocks in a stock market. We will introduce several interest rate models and use them to price various derivatives defined on the underlying assets. Note that each fixed-income asset or derivative may have its own maturity time.

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$. Assume $\{r_t\}$ is an adapted process with $P(\int_0^T |r_t| \, dt < \infty) = 1$. The filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ is usually generated by a multidimensional Brownian motion (called shocks or factors) as a source of uncertainty. Since our main purpose is derivative pricing, we assume the existence of a risk neutral measure $Q$, equivalent to $P$ following the Girsanov transformation (detail skipped), under which the discounted processes of all underlying assets are martingales. This will rule out arbitrage opportunities, according to the (continuous-time) Fundamental Theorem of Asset Pricing, which we have not presented. For $0 \leq t < \tau \leq T$, three equivalent term structures can be defined as follows:

- **Zero-coupon bond:** Let $B(t, \tau)$ be the time-$t$ price of a bond with maturity $\tau$ (called $\tau$-bond) and the par value $B(\tau, \tau) = 1$. The risk-neutral valuation principle implies
  \[
  B(t, \tau) = \mathbb{E}_Q \left[ \exp \left( -\int_t^\tau r_u du \right) \bigg| \mathcal{F}_t \right].
  \]

- **Yield** $Y(t, \tau)$, defined by
  \[
  Y(t, \tau) = \frac{-1}{\tau - t} \log B(t, \tau),
  \]
  which follows from
  \[
  B(t, \tau) \exp[Y(t, \tau)(\tau - t)] = 1,
  \]
  i.e. $Y(t, \tau)$ is the constant compounding interest rate that brings $B(t, \tau)$ to the par value.

- **Forward rate** $f(t, \tau)$, defined by
  \[
  f(t, \tau) = -\frac{\partial}{\partial \tau} \log B(t, \tau),
  \]
  which follows from
  \[
  B(t, \tau) \exp \left[ \int_t^\tau f(t, u) du \right] = 1,
  \]
i.e. $f(t, \tau)$ is the instantaneous rate locked in at $t$ for risk-free borrowing or lending at $\tau$, with $f(t, t) = r_t$. Intuitively, $f(t, \tau)$ should be interpreted as the interest rate over the infinitesimal time interval $[\tau, \tau + \Delta\tau]$ as seen at time $t$.

To ease the notation, assume $0 \leq \tau \leq T$ and consider a European option $g(B(\tau, T))$ with maturity $\tau$ written on a zero-coupon bond with maturity equal to the horizon $T$. The option $g$ can be hedged by a dynamic portfolio $h = \{(h_{0t}, h_{1t}), \ t \in [0, T]\}$, where $h_{0t}B_t$ and $h_{1t}B(t, T)$ represent the time-$t$ balance of the bank account and the time-$t$ holding value of the zero-coupon bond respectively. The time-$t$ value of the portfolio is

$$V_t = h_{0t}B_t + h_{1t}B(t, T),$$

with the corresponding self-financing condition

$$dV_t = h_{0t} dB_t + h_{1t} dB(t, T).$$

Under certain regularity conditions, the time-$t$ value of the option $g$ is given by

$$V_t = EQ\left[ \exp\left(-\int_0^\tau r_u du\right) g(B(\tau, T)) \bigg| \mathcal{F}_t \right],$$

which can be calculated in each problem with a specified interest rate model.

### 19.2 Three classical models

We will present three continuous-time single factor interest rate models in this section, whose discrete-time versions appeared in Lecture 8. In each model, the SDE represents the risk neutral dynamics under $Q$.

#### 19.2.1 Vasicek model

Assume the short rate under $Q$ satisfies the SDE

$$dr_t = a(b - r_t) \ dt + \sigma \ dW_t$$

with positive constant parameters $a, b$ and $\sigma$. Note that $X = \{X_t\}$ with $X_t = r_t - b$ is an Ornstein-Uhlenbeck (O-U) process satisfying the SDE

$$dX_t = -aX_t \ dt + \sigma \ dW_t.$$  

Hence $X$ is a Gaussian process, which implies that $Q(r_t < 0) > 0$ for every $t \in [0, T]$ — an unreasonable behavior in practice. Pricing bond options would be easy due to nice properties of the O-U process.
19.2.2 Cox-Ingersoll-Ross (CIR) model

The short rate process under \( Q \) follows

\[
dr_t = a(b - r_t) \, dt + \sigma \sqrt{r_t} \, dW_t
\]

with constant parameters \( a, b \) and \( \sigma \). Suppose \( r_0 = x > 0 \) and define

\[
T_{x,0} = \inf\{ t > 0 : r_t = 0 \}, \text{ or } T_{x,0} = \infty \text{ if } r_t > 0 \forall \, t > 0.
\]

The following proposition shows that the CIR model is an important improvement of the Vasicek model.

**Proposition 19.1** There are three cases based on different ranges of the parameters:

- **Case 1** If \( ab \geq \sigma^2/2 \), then \( Q(T_{x,0} = \infty) = 1 \forall \, x > 0 \).
- **Case 2** If \( 0 \leq ab < \sigma^2/2 \) and \( a \geq 0 \), then \( Q(T_{x,0} < \infty) = 1 \forall \, x > 0 \).
- **Case 3** If \( 0 \leq ab < \sigma^2/2 \) and \( a < 0 \), then \( 0 < Q(T_{x,0} < \infty) < 1 \forall \, x > 0 \).

Proposition 19.1 provides us with a guideline for setting the parameters \( a, b, \sigma \). \( \{r_t\} \) in (19.5) is an example of Feller’s square-root diffusions which enjoy great popularity in financial economics. It is a Markov process with a closed-form transition density: the conditional density of \( r_u \) given \( r_t \) for \( t < u \) is a scaled version of non-central \( \chi^2 \) distribution. Prices of the zero-coupon bond and some bond options under a CIR model can be obtained with explicit expressions.

19.2.3 Heath-Jarrow-Morton (HJM) model

The Vasicek and CIR models, although very simple, do not model term structures directly. When market term structure data \( \{B(t, \tau), \tau \in [t, T]\} \) are observed at time \( t \), they are inconsistent with the output generated by these two short rate models. In contrast, HJM proposes a general mechanism that models term structures directly and takes observed term structures as inputs.

\[
df(t, \tau) = \alpha(t, \tau) \, dt + \sigma(t, \tau) \, dW_t
\]

given the initial term structure \( f(0, \tau) = g(\tau), \tau \in [0, T] \), where \( \alpha(\cdot), \sigma(\cdot) \) satisfy the HJM drift condition

\[
\alpha(t, \tau) = \sigma(t, \tau) \int_t^\tau \sigma(t, u) \, du
\]
to avoid arbitrage.

The integral form of (19.6)

\[ f(t, \tau) = g(\tau) + \int_0^t \alpha(u, \tau) \, du + \int_0^t \sigma(u, \tau) \, dW_u, \]  

(19.8)

along with \( f(t, t) = r_t \), will give us

\[ r_t = f(0, t) + \int_0^t \alpha(u, t) \, du + \int_0^t \sigma(u, t) \, dW_u. \]  

(19.9)

Note that in a general HJM model, the drift coefficient \( \alpha(t, \tau) \) and volatility coefficient \( \sigma(t, \tau) \) are not just functions of “current state” \( f(t, \tau) \), they also involve the history \( \{ f(u, \tau), 0 < u < t \} \). For fixed \( \tau \), the process \( \{ f(t, \tau), 0 \leq t \leq \tau \} \) is non-Markovian. Therefore, the computation required for the HJM model is extremely complicated, as a price to pay for greater modeling flexibility than the Vasicek and CIR models.

Multi-factor versions of Vasicek, CIR and HJM are available that further improve the results in empirical studies.