Lecture 8  Some Spot Rate and Term Structure Models

8.1 Spot rate models

Different expressions of the drift $\mu$ and volatility $\sigma$ in the SDE (7.14)

$$\Delta r(t+1) = \mu(t, r(t)) + \sigma(t, r(t)) \epsilon_t$$

lead to various spot rate models. In this lecture, we first consider time-homogeneous SDEs in which

$$\mu(t, r(t)) = \mu(r(t)) \quad \text{and} \quad \sigma(t, r(t)) = \sigma(r(t)),$$

i.e. the drift and volatility do not involve time $t$ explicitly. We then study an inhomogeneous SDE in Hull-White model, along with Markov chains represented by binomial or trinomial trees.

8.1.1 Vasicek and CIR models

Consider the following special cases in which $a, b, \sigma$ are all positive constants, and $\beta$ is a constant with $0 \leq \beta \leq 1$.

**Example 8.1 (Vasicek model)**  \[ \mu(t, r(t)) = a[b - r(t)], \quad \sigma(t, r(t)) = \sigma. \]

**Example 8.2 [Cox-Ingersoll-Ross (CIR) model]**  \[ \mu(t, r(t)) = a[b - r(t)], \quad \sigma(t, r(t)) = \sigma \sqrt{r(t)}. \]

**Example 8.3 (a more general class)**  \[ \mu(t, r(t)) = a[b - r(t)], \quad \sigma(t, r(t)) = \sigma [r(t)]^\beta. \]

A common feature of these models is the mean reversion property: the spot rate $r$ tends to decrease if $r > b$ and increase if $r < b$. An advantage of CIR over Vasicek is its capability of enforcing positive values on $r$. To formulate the question more precisely, assume $r(1) = x > 0$ and let

$$\tau_x = \inf \{2 \leq t \leq T : r(t) \leq 0\}. \quad (8.1)$$

The question is: under what conditions on the parameters $a, b$ and $\sigma$ in the SDEs for Vasicek and CIR respectively, $Q(\tau_x \leq T) = 0$ will be satisfied? Note that $Q(\tau_x \leq T) = 0$ and $P(\tau_x \leq T) = 0$ are equivalent since $Q$ is an EMM. Both imply that the values of $r$ remain positive in the whole process. It turns out that those conditions needed in CIR are quite mild and reasonable, but the conditions needed in Vasicek are too strict to be realistic. Further discussion on various modeling issues will appear in continuous-time finance.
8.1.2 Hull-White model

In a very general form, Hull-White model can be expressed via the SDE

$$\Delta r(t+1) = a(t) [b(t) - r(t)] + \sigma(t) [r(t)]^\beta \epsilon_t,$$  \hfill (8.2)

where $\beta \geq 0$ is still a constant, but $a(t)$, $b(t)$ and $\sigma(t)$ are positive-valued deterministic functions. These functions greatly enhance modeling flexibility, e.g. the initial term structure can be incorporated into $a(t)$ and $b(t)$. The condition on $\{\epsilon_t\}$ is often relaxed in this model, e.g. $\epsilon_t$’s are independent with $Q(\epsilon_t = 1) = q(t) < 1$.

Under what conditions on functions $a$, $b$ and $\sigma$, an underlying MC $X$ can be represented by a recombining tree such that $r$ is defined by $r(t+1) = g_t(n_t)$ (Q1 in Lecture 7)? A recombining tree requires that starting from any node, a “up-down” combination ($\epsilon_t^1 = 1$ and $\epsilon_{t+1} = -1$) and a “down-up” combination ($\epsilon_t = -1$ and $\epsilon_{t+1} = 1$) should merge at the same node. In other words, starting from $r(t)$, these two combinations must arrive at the same value of $r(t+2)$. Standard calculation turns this into the condition

$$\sigma(t+1) = [1 - a(t+1)] \sigma(t) \quad \forall \ t,$$  \hfill (8.3)

which also requires $a(t) < 1 \ \forall t$ (for positive volatility).

Trinomial trees are often used in this situation to provide an extra degree of freedom. In a trinomial tree, three branches come out of each node: up, middle and down. There are $2t+1$ nodes at time $t$. The recombining assumption consists of “up-down = middle-middle = down-up”, “up-middle = middle-up” and “down-middle = middle-down”. The three possible moves up, middle and down correspond to $\epsilon_t = 1$, $\epsilon_t = 0$ and $\epsilon_t = -1$, with risk neutral probabilities $Q(\epsilon_t = 1) = Q(\epsilon_t = -1) = q(t) < 1/2$ and $Q(\epsilon_t = 0) = 1 - 2q(t)$. The same argument shows that the recombining condition is still given by (8.3).

8.1.3 Refined lattice and SDE

So far, the binomial (or trinomial tree) models and SDEs for spot rates have fixed integer 1 as the span for both time increment and space increment. Refinement of this is needed for more realistic models, especially for passing the limit to continuous-time finance. To carry out this technical step, fix $T > 0$ and positive integer $n$, let $\delta = T/n$ be the time increment and $\sqrt{\delta}$ the space increment in the binomial or trinomial trees. Accordingly, for $j = 0, 1, \ldots, n-1$, we let

$$r_\delta(j) = r(j\delta), \quad \Delta r_\delta(j+1) = r_\delta(j+1) - r_\delta(j).$$
The functions $\mu_\delta(j, r_\delta(j))$, $\sigma_\delta(j, r_\delta(j))$, $a_\delta(j)$ and $b_\delta(j)$ are defined in the same way. The SDE (7.14) becomes

$$\Delta r_\delta(j + 1) = \mu_\delta(j, r_\delta(j)) \delta + \sigma_\delta(j, r_\delta(j)) \sqrt{\delta} \epsilon_j.$$  

(8.4)

Keep in mind that for a fixed $\delta$ (or $n$), the basic setting in Lecture 2 is still valid. However, if we let $n \to \infty$ (thus $\delta \to 0$), many complicated technical issues will need to be taken care of.

### 8.2 Term structure models

Sections 7.3, 7.4 and 8.1 focused on modeling the spot rate $r$. The corresponding term structures can be derived based on $r$. This method is relatively easy to implement. A drawback is that it lacks the capability of modeling yields (or zero-coupon bond prices) with different maturities. For example, if one is interested in the spread between short and long term interest rates, those spot rate models would have difficulties to model such a feature explicitly.

An alternative approach, proposed by Heath, Jarrow and Morton (HJM), is to consider an entire yield curve as a state variable that evolves over time. In principle, one can model any one of the three equivalent term structures: zero-coupon bond prices, yields and forward rates. It is often convenient to work with forward rates. In this section, we start with the basic notation in HJM set-up, followed by more detailed discussion on a special case of HJM — the Ho-Lee model, and show how some spot rate models are derived from Ho-Lee model.

#### 8.2.1 HJM setting

Following the basic framework in Lecture 7, we extend the idea of binomial trees to a state space consisting of yield curves. A given value of the time $t$ term structure $\{f(t, \tau), t \leq \tau \leq T - 1\}$ may move “up” with a (risk neutral) probability $q$ or “down” with probability $1 - q$ to one of the two possible values of the time $t + 1$ term structure $\{f(t + 1, \tau), t + 1 \leq \tau \leq T - 1\}$. In general, $q$ may depend on $t$ and the value of time $t$ state. Such a binomial tree need not be recombining. To fix the notation for a binomial tree in this context, a typical node is denoted by $(t, k)$, where $t = 0, 1, \ldots, T - 1; k = 0, 1, \ldots, 2^t - 1$. Let $f_{tk}$ be a possible value of the time $t$ term structure, which is a vector with components $f_{tk}(\tau), t \leq \tau \leq T - 1$. In particular, $f_{t+1,2k}$ and $f_{t+1,2k+1}$ are the two immediate “descendants” of $f_{tk}$ in the binomial tree. The risk neutral (conditional) probabilities moving from the $(t, k)$ to the next generation $(t + 1, 2k)$ or $(t + 1, 2k + 1)$ are $q_{tk}$ and $1 - q_{tk}$ respectively.

**Example 8.4** We use the data in Example 7.1 to illustrate the above notation. First, Table 8.1 contains forward rates calculated from Table 7.1 by using the formula (7.4).
Table 8.1: Forward rates in Example 7.1.

Next, the values \{f_{tk}\} of the term structures are given as follows:

\[
f_{00} = (f(0, 0), f(0, 1), f(0, 2)) \text{ on } \{\omega_1, \omega_2, \omega_3, \omega_4\} = (0.04, 0.045, 0.044)
\]

\[
f_{10} = (f(1, 1), f(1, 2)) \text{ on } \{\omega_1, \omega_2\} = (0.06, 0.06)
\]

\[
f_{11} = (f(1, 1), f(1, 2)) \text{ on } \{\omega_3, \omega_4\} = (0.03, 0.03)
\]

\[
f_{20} = f(2, 2) \text{ on } \{\omega_1\} = 0.07
\]

\[
f_{21} = f(2, 2) \text{ on } \{\omega_2\} = 0.05
\]

\[
f_{22} = f(2, 2) \text{ on } \{\omega_3\} = 0.04
\]

\[
f_{23} = f(2, 2) \text{ on } \{\omega_4\} = 0.02
\]

There are two restrictions that a model builder must keep in mind in the specification of term structures of forward rates: positivity of \(f(\cdot, \cdot)\) and no arbitrage. It follows from (7.5), (7.8) and (7.11) that for \(\tau = t + 2, \ldots, T\),

\[
\prod_{s=t+2}^{\tau} [1 + f(t, s - 1)]^{-1} = E_Q \left[ \prod_{s=t+2}^{\tau} [1 + f(t + 1, s - 1)]^{-1} \right] \bigg| \mathcal{F}_t. \tag{8.5}
\]

If the left-hand side is the value at a node \((t, k)\), say, then we have

\[
\prod_{s=t+2}^{\tau} \frac{1}{1 + f_{tk}(s-1)} = \frac{1}{q_{tk} \prod_{s=t+2}^{\tau} \frac{1}{1 + f_{t+1,2k}(s-1)}} + \frac{1 - q_{tk}}{1 + f_{t+1,2k+1}(s-1)} \prod_{s=t+2}^{\tau} \frac{1}{1 + f_{t+1,2k+1}(s-1)} \tag{8.6}
\]

Example 8.5  For given \(f(t, t), \ldots, f(t, T-1)\), consider in each of the following three cases whether it is possible to specify positive values of \(f_{t+1,2k+1}(t+1), \ldots, f_{t+1,2k+1}(T-1)\) such that (8.6) holds.
Case 1. Positive values $f_{t+1,2k}(t+1), \ldots, f_{t+1,2k}(T-1)$ and $q_{tk} \in (0,1)$ are fixed;

Case 2. $q_{tk} \in (0,1)$ is fixed, but there is flexibility in choosing positive values $f_{t+1,2k}(t+1), \ldots, f_{t+1,2k}(T-1)$.

Case 3. Positive values $f_{t+1,2k}(t+1), \ldots, f_{t+1,2k}(T-1)$ are fixed, but there is flexibility in choosing $q_{tk} \in (0,1)$.

8.2.2 Ho-Lee model

The paper of Ho-Lee (Journal of Finance, 1986) proposed the first yield curve model, based on which the general HJM was developed later on. In what follows, we first present Ho-Lee model for the term structures of zero-coupon bond prices then turn it into the equivalent version for forward rates, which enables us to see Ho-Lee model explicitly as a special case of HJM setting in Subsection 8.2.1 and to connect Ho-Lee model to some short rate models.

A simple feature of Ho-Lee model is that it can be represented by a recombining binomial tree (a general HJM tree need not be recombining). Therefore, the notation in Subsection 8.2.1 is modified to be adapted to the triangular lattice in Figure 7.2.

- On the triangle lattice $\{(t,k) : k = 0,1,\ldots,t; t = 0,1,\ldots,T\}$, the time $t$ term structure $\{B(t,\tau), t < \tau \leq T\}$ assumes values $\{B_{tk}(\tau), t < \tau \leq T\}$.

- For $t < \tau \leq T$, let $s = \tau - t$ be the time to maturity, $1 \leq s \leq T - t$. A perturbation factor $\eta(s)$ takes either the value $u(s) > 1$ (an “inflation” factor or a “up” move) with probability $q$, or the value $0 < d(s) < 1$ (a “deflation” factor or a “down” move) with probability $1-q$, where the (risk neutral) probability $q$ is assumed to be a constant. The conditions

$$\eta(0) = 1 \quad (8.7)$$

and

$$q u(s) + (1-q) d(s) = 1 \quad \forall \ 1 \leq s \leq T \quad (8.8)$$

are satisfied.

- The evolution of time $t$ term structure in $(t,t+1]$ follows

$$B(t+1,\tau) = B(t,\tau) \frac{B_{tk}(\tau)}{B_{tk}(t+1)} \eta(\tau - t - 1), \quad (8.9)$$

which can be spelled out as

$$B_{t+1,k+1}(\tau) = B_{tk}(\tau) \frac{B_{t+1,k+1}(\tau)}{B_{tk}(t+1)} \ u(s - 1), \quad (8.10)$$
or
\[
B_{t+1,k}(\tau) = \frac{B_{tk}(\tau)}{B_{tk}(t+1)} d(s-1).
\] (8.11)

- For \( t + 2 \leq \tau \), a “up-down” combination gives rise to
\[
B_{t+2,k+1}(\tau) = \frac{B_{tk}(\tau)}{B_{tk}(t+2)} \frac{u(s-1) d(s-2)}{u(1)};
\] (8.12)

while a “down-up” combination gives rise to
\[
B_{t+2,k+1}(\tau) = \frac{B_{tk}(\tau)}{B_{tk}(t+2)} \frac{d(s-1) u(s-2)}{d(1)}.
\] (8.13)

Therefore, the following “path-independent” condition that forces a recombining binomial tree is needed:
\[
u(s-1) d(s-2) d(1) = d(s-1) u(s-2) u(1).
\] (8.14)

Using (8.8), we can simplify (8.14) as
\[
\frac{1}{u(s-1)} = \frac{\theta}{u(s-2)} + q(1-\theta) \quad \forall \ 2 \leq s \leq T,
\] (8.15)

where \( \theta \) is a constant, called the interest rate spread and given by
\[
q + (1-q)\theta = \frac{1}{u(1)},
\] (8.16)

thus \( 0 < \theta < 1 \). It follows from the induction and the condition \( u(0) = 1 \) that for \( 0 \leq s \leq T \),
\[
u(s) = \frac{1}{q + (1-q)\theta^s},
\] (8.17)

and
\[
d(s) = \frac{\theta^s}{q + (1-q)\theta^s}.
\] (8.18)

- To sum up, the dynamics of term structures in Ho-Lee model is determined by two constant parameters \( q \) and \( \theta \), which makes Ho-Lee model computationally efficient but also too restrictive (underparameterized). For instance, the zero-coupon bond price may not always fall in the range \([0, 1]\). Still, Ho-Lee model plays a significant role in modeling the entire yield curve. The state at node \((t, k)\) (the time \( t \) yield curve) \((B_{tk}(t+1), B_{tk}(t+2), \ldots, B_{tk}(T))\) will change to \((B_{t+1,k+1}(t+1), B_{t+1,k+1}(t+2), \ldots, B_{t+1,k+1}(T))\) at node \((t+1, k+1)\) with probability \( q \), or \((B_{t+1,k}(t+1), \ldots, B_{t+1,k}(T))\) at node \((t+1, k)\) with probability \( 1-q \).
Note the relation

\[
\frac{1}{1 + f(t, \tau)} = \frac{B(t, \tau + 1)}{B(t, \tau)}. \tag{8.19}
\]

Following (8.10), (8.11), (8.17) and (8.18), the dynamics of forward rates is given by

\[
\frac{1}{1 + f_{t+1,k+1}(\tau)} = \frac{1}{1 + f_{tk}(\tau)} \frac{q + (1 - q)\theta^{\tau-t-1}}{q + (1 - q)\theta^{\tau-t}}, \tag{8.20}
\]

or

\[
\frac{1}{1 + f_{t+1,k}(\tau)} = \frac{1}{1 + f_{tk}(\tau)} \frac{\theta \left[q + (1 - q)\theta^{\tau-t-1}\right]}{q + (1 - q)\theta^{\tau-t}}. \tag{8.21}
\]

The state \((f_{tk}(t), f_{tk}(t+1), \ldots, f_{tk}(T-1))\) will turn to \((f_{t+1,k+1}(t+1), \ldots, f_{t+1,k+1}(T-1))\) with probability \(q\), or \((f_{t+1,k}(t+1), \ldots, f_{t+1,k}(T-1))\) with probability \(1 - q\). Compare this to Subsection 8.2.1, it can be seen clearly that Ho-Lee is a special case of HJM. In particular, Ho-Lee corresponds to a recombining binomial tree, and assumes a constant probability \(q\) at every node.

### 8.2.3 Spot rate models from Ho-Lee

In a binomial tree associated with Ho-Lee model, the recombining feature implies that the total number of “up’s” (or “down’s”) determines the dynamics, regardless of their orders. Iterating (8.20) and (8.21) in \(t\) leads to

\[
\frac{1}{1 + f_{t,n_t}(\tau)} = \frac{\theta^{t-n_t}}{1 + f(0, \tau)} \frac{q + (1 - q)\theta^{\tau-t}}{q + (1 - q)\theta^{\tau-t}}, \tag{8.22}
\]

where \(n_t\) represents the total number of “up’s” up to \(t\). Set \(\tau = t\) and recall (7.8), we have

\[
r(t + 1) = [1 + f(0, t)] (q\theta^{-t} + 1 - q) \theta^{n_t} - 1. \tag{8.23}
\]

The only random variable in (8.23) is \(n_t\) which follows a binomial distribution.